Seminari 6

Matematika za ekonomiste 2

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Oznake

- Funkcija dvije varijable: $z = z(x, y)$
- Parcijalna derivacija po varijabli x

$$z_x \qquad z'_x \qquad \frac{\partial z}{\partial x}$$

- Parcijalna derivacija po varijabli y

$$z_y \qquad z'_y \qquad \frac{\partial z}{\partial y}$$

Zadatak 1

$(x^n)' = nx^{n-1}$

$(cu)'(x) = c \cdot u'(x)$

$\left(\sqrt{x}\right)' = \dfrac{1}{2\sqrt{x}}$

Odredite parcijalne derivacije sljedećih funkcija:

a) $f(x, y) = x^2 + y^2$

b) $g(x, y) = 3x^2 + xy + \sqrt{y}$

c) $z = \dfrac{y}{x}$

$z = yx^{-1}$

Rješenje

a) $f_x = 2x + 0 = 2x \qquad f_y = 0 + 2y = 2y$

b) $g_x = 6x + y + 0 = 6x + y \qquad g_y = 0 + x + \dfrac{1}{2\sqrt{y}} = x + \dfrac{1}{2\sqrt{y}}$

c) $z_x = y \cdot \left(-x^{-2}\right) = -\dfrac{y}{x^2} \qquad z_y = x^{-1} \cdot 1 = \dfrac{1}{x}$

Zadatak 2

$(uv)'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$

Odredite parcijalne derivacije sljedećih funkcija:

a) $z = xe^y$ b) $z = ye^y + \sqrt{x}$ c) $u(x, y) = \dfrac{2x - y}{x + y}$

$\left(\dfrac{u}{v}\right)'(x) = \dfrac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v(x)^2}$

Rješenje

$(cu)'(x) = c \cdot u'(x)$

a) $z_x = e^y \cdot 1 = e^y \qquad z_y = x \cdot e^y = xe^y$

b) $z_x = 0 + \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x}} \qquad z_y = 1 \cdot e^y + y \cdot e^y + 0 = (1 + y)e^y$

$(e^x)' = e^x$

$\left(\sqrt{x}\right)' = \dfrac{1}{2\sqrt{x}}$

c) $u_x = \dfrac{2 \cdot (x + y) - (2x - y) \cdot 1}{(x + y)^2} = \dfrac{3y}{(x + y)^2}$

$u_y = \dfrac{-1 \cdot (x + y) - (2x - y) \cdot 1}{(x + y)^2} = \dfrac{-3x}{(x + y)^2}$

Zadatak 3

Odredite parcijalne derivacije sljedećih funkcija:

a) $z(x,y) = (x+2y)e^{x^2+y^3}$

b) $z = \dfrac{x}{\sqrt{x^2+y^2}}$

c) $z = 2^{\sin\frac{y}{x}}$

d) $z = x^y$

Rješenje

$\boxed{(e^x)' = e^x}$ $\boxed{(e^{\text{nešto}})' = e^{\text{nešto}} \cdot (\text{nešto})'}$

a)

$$z_x = (x+2y)'_x \cdot e^{x^2+y^3} + (x+2y)\cdot\left(e^{x^2+y^3}\right)'_x =$$

$$= 1 \cdot e^{x^2+y^3} + (x+2y)e^{x^2+y^3} \cdot (x^2+y^3)'_x =$$

$$= e^{x^2+y^3} + (x+2y)e^{x^2+y^3} \cdot 2x =$$

$$= e^{x^2+y^3} + (2x^2+4xy)e^{x^2+y^3} = (2x^2+4xy+1)e^{x^2+y^3}$$

$$z_y = (x+2y)'_y \cdot e^{x^2+y^3} + (x+2y)\cdot\left(e^{x^2+y^3}\right)'_y =$$

$$= 2 \cdot e^{x^2+y^3} + (x+2y)e^{x^2+y^3} \cdot (x^2+y^3)'_y =$$

$$= 2e^{x^2+y^3} + (x+2y)e^{x^2+y^3} \cdot 3y^2 =$$

$$= 2e^{x^2+y^3} + (3xy^2+6y^3)e^{x^2+y^3} = (3xy^2+6y^3+2)e^{x^2+y^3}$$

$\boxed{(uv)'(x) = u'(x)\cdot v(x) + u(x)\cdot v'(x)}$ $\boxed{z = (x+2y)e^{x^2+y^3}}$

$\boxed{\left(\sqrt{\text{nešto}}\right)' = \dfrac{1}{2\sqrt{\text{nešto}}} \cdot (\text{nešto})'}$ $\boxed{z = \dfrac{x}{\sqrt{x^2+y^2}}}$ $\boxed{\left(\sqrt{x}\right)' = \dfrac{1}{2\sqrt{x}}}$

b)

$$z_x = \frac{(x)'_x \cdot \sqrt{x^2+y^2} - x \cdot \left(\sqrt{x^2+y^2}\right)'_x}{\sqrt{x^2+y^2}^{\,2}} =$$

$$= \frac{1 \cdot \sqrt{x^2+y^2} - x \cdot \dfrac{1}{2\sqrt{x^2+y^2}} \cdot (x^2+y^2)'_x}{x^2+y^2} =$$

$$= \frac{\sqrt{x^2+y^2} - \dfrac{x}{2\sqrt{x^2+y^2}} \cdot 2x}{x^2+y^2} = \frac{\sqrt{x^2+y^2} - \dfrac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} =$$

$$= \frac{x^2+y^2-x^2}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$\boxed{\left(\dfrac{u}{v}\right)'(x) = \dfrac{u'(x)\cdot v(x) - u(x)\cdot v'(x)}{v(x)^2}}$

$\boxed{\left(\sqrt{\text{nešto}}\right)' = \dfrac{1}{2\sqrt{\text{nešto}}} \cdot (\text{nešto})'}$ $\boxed{z = \dfrac{x}{\sqrt{x^2+y^2}}}$ $\boxed{\left(\sqrt{x}\right)' = \dfrac{1}{2\sqrt{x}}}$

$$z_y = \frac{(x)'_y \cdot \sqrt{x^2+y^2} - x \cdot \left(\sqrt{x^2+y^2}\right)'_y}{\sqrt{x^2+y^2}^{\,2}} =$$

$$= \frac{0 \cdot \sqrt{x^2+y^2} - x \cdot \dfrac{1}{2\sqrt{x^2+y^2}} \cdot (x^2+y^2)'_y}{x^2+y^2} =$$

$$= \frac{-\dfrac{x}{2\sqrt{x^2+y^2}} \cdot 2y}{x^2+y^2} = \frac{\dfrac{-xy}{\sqrt{x^2+y^2}}}{x^2+y^2} =$$

$$= \frac{-xy}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}$$

$\boxed{\left(\dfrac{u}{v}\right)'(x) = \dfrac{u'(x)\cdot v(x) - u(x)\cdot v'(x)}{v(x)^2}}$

$(a^x)' = a^x \ln a$ $(a^{\text{nešto}})' = a^{\text{nešto}} \ln a \cdot (\text{nešto})'$ $z = 2^{\sin \frac{y}{x}}$

$(x^n)' = nx^{n-1}$ $(\sin x)' = \cos x$ $(\sin(\text{nešto}))' = \cos(\text{nešto}) \cdot (\text{nešto})'$

c) $$z_x = 2^{\sin \frac{y}{x}} \ln 2 \cdot \left(\sin \frac{y}{x}\right)'_x = 2^{\sin \frac{y}{x}} \ln 2 \cdot \cos \frac{y}{x} \cdot \left(\frac{y}{x}\right)'_x =$$

$$= 2^{\sin \frac{y}{x}} \ln 2 \cdot \cos \frac{y}{x} \cdot \frac{-y}{x^2} = -\frac{y}{x^2} \cdot 2^{\sin \frac{y}{x}} \ln 2 \cdot \cos \frac{y}{x}$$

$$z_y = 2^{\sin \frac{y}{x}} \ln 2 \cdot \left(\sin \frac{y}{x}\right)'_y = 2^{\sin \frac{y}{x}} \ln 2 \cdot \cos \frac{y}{x} \cdot \left(\frac{y}{x}\right)'_y =$$

$$= 2^{\sin \frac{y}{x}} \ln 2 \cdot \cos \frac{y}{x} \cdot \frac{1}{x} = \frac{1}{x} \cdot 2^{\sin \frac{y}{x}} \ln 2 \cdot \cos \frac{y}{x}$$

$z = x^y$

d) $z_x = yx^{y-1}$ $z_y = x^y \ln x$

Parcijalne derivacije drugog reda – oznake

- Funkcija dvije varijable: $z = z(x, y)$

z_{xx}	z''_{xx}	$\frac{\partial^2 z}{\partial x^2}$
z_{xy}	z''_{xy}	$\frac{\partial^2 z}{\partial x \partial y}$
z_{yx}	z''_{yx}	$\frac{\partial^2 z}{\partial y \partial x}$
z_{yy}	z''_{yy}	$\frac{\partial^2 z}{\partial y^2}$

Zadatak 4

$(e^{\text{nešto}})' = e^{\text{nešto}} \cdot (\text{nešto})'$ $(e^x)' = e^x$

Izračunajte vrijednosti parcijalnih derivacija funkcije

$(\ln x)' = \frac{1}{x}$

$$f(x, y, z) = e^{2xz} - \ln(yz) + 1$$

u točki $(0, 2, 1)$.

$(\ln(\text{nešto}))' = \frac{1}{\text{nešto}} \cdot (\text{nešto})'$

Rješenje

$$f_x = e^{2xz} \cdot (2xz)'_x - 0 + 0 = e^{2xz} \cdot 2z = 2ze^{2xz}$$

$$f_y = 0 - \frac{1}{yz} \cdot (yz)'_y + 0 = -\frac{1}{yz} \cdot z = -\frac{1}{y}$$

$$f_z = e^{2xz} \cdot (2xz)'_z - \frac{1}{yz} \cdot (yz)'_z + 0 = e^{2xz} \cdot 2x - \frac{1}{yz} \cdot y = 2xe^{2xz} - \frac{1}{z}$$

$$f_x(\overset{x}{0}, \overset{y}{2}, \overset{z}{1}) = 2 \cdot 1 \cdot e^{2 \cdot 0 \cdot 1} = 2e^0 = 2 \qquad f_y(\overset{x}{0}, \overset{y}{2}, \overset{z}{1}) = -\frac{1}{2}$$

$$f_z(\overset{x}{0}, \overset{y}{2}, \overset{z}{1}) = 2 \cdot 0 \cdot e^{2 \cdot 0 \cdot 1} - \frac{1}{1} = 0 \cdot e^0 - 1 = -1$$

Zadatak 5

Odredite parcijalne derivacije drugog reda funkcije $z(x, y) = y^2 \cdot 2^x$.

Rješenje

$(a^x)' = a^x \ln a$

$(x^n)' = nx^{n-1}$

$$z_x = y^2 \cdot 2^x \ln 2$$

$$z_y = 2^x \cdot 2y = y \cdot 2^{x+1}$$

$$z_{xx} = (z_x)_x = y^2 \ln 2 \cdot 2^x \ln 2 = y^2 \cdot 2^x \ln^2 2$$

$$z_{xy} = (z_x)_y = 2^x \ln 2 \cdot 2y = y \cdot 2^{x+1} \ln 2$$

$$z_{yx} = (z_y)_x = y \cdot 2^{x+1} \ln 2 \cdot (x+1)'_x = y \cdot 2^{x+1} \ln 2$$

$$z_{yy} = (z_y)_y = 2^{x+1} \cdot 1 = 2^{x+1}$$

$(uv)'(x) = u'(x)\cdot v(x) + u(x)\cdot v'(x)$

Zadatak 6

Zadana je funkcija $f(x,y,z) = z\cdot y^x$. *Odredite* $\dfrac{\partial^3 f}{\partial x \partial y \partial z}$.

Rješenje

$\dfrac{\partial^3 f}{\partial x \partial y \partial z} \rightarrow f_{xyz}$

$$f_x = z\cdot y^x \ln y = zy^x \ln y$$

$$f_{xy} = (f_x)_y = zxy^{x-1}\cdot \ln y + zy^x \cdot \frac{1}{y} = xzy^{x-1}\ln y + zy^{x-1}$$

$$f_{xyz} = (f_{xy})_z = xy^{x-1}\ln y + y^{x-1}$$

$(a^x)' = a^x \ln a$ $\quad (x^n)' = nx^{n-1}$ $\quad (\ln x)' = \dfrac{1}{x}$

Zadatak 7

Odredite lokalne ekstreme funkcije $f(x,y) = 80 - y^2 + x^3 + 12y - 3x$.

Rješenje

$$\begin{array}{l|l} f_x = 3x^2 - 3 & 3x^2 - 3 = 0 \longrightarrow x^2 = 1 \longrightarrow x_1 = 1,\ x_2 = -1 \\ f_y = -2y + 12 & -2y + 12 = 0 \longrightarrow y = 6 \end{array}$$

Stacionarne točke: $(\overset{x_1}{1}, \overset{y}{6}),\ (\overset{x_2}{-1}, \overset{y}{6})$

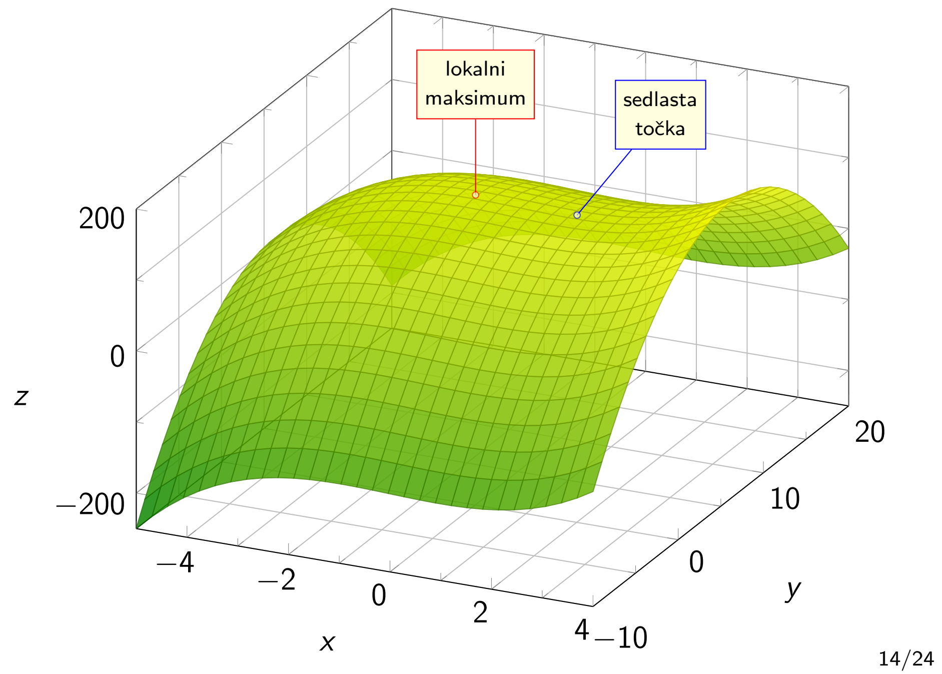
$$H(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 0 \\ 0 & -2 \end{vmatrix}$$

$f_{xx} = 6x, \quad f_{xy} = 0, \quad f_{yy} = -2$

$$H(\overset{x}{1}, \overset{y}{6}) = \begin{vmatrix} 6 & 0 \\ 0 & -2 \end{vmatrix} = -12 < 0 \longrightarrow \text{sedlasta točka}$$

$$H(\overset{x}{-1}, \overset{y}{6}) = \begin{vmatrix} \overset{<0}{-6} & 0 \\ 0 & -2 \end{vmatrix} = 12 > 0 \longrightarrow \text{točka lokalnog maksimuma}$$

$$f(\overset{x}{-1}, \overset{y}{6}) = 80 - 6^2 + (-1)^3 + 12\cdot 6 - 3\cdot(-1) = 118$$



Zadatak 8

Odredite lokalne ekstreme funkcije

$$z(x,y) = \frac{8}{x} + \frac{x}{y} + y.$$

Rješenje

$$z(x,y) = 8x^{-1} + xy^{-1} + y$$

- $x \neq 0,\ y \neq 0$

$$z_x = -8x^{-2} + y^{-1} \qquad -8x^{-2} + y^{-1} = 0$$

$$z_y = -xy^{-2} + 1 \qquad -xy^{-2} + 1 = 0$$

$$\begin{aligned}-8x^{-2}+y^{-1}&=0\\ -xy^{-2}+1&=0\end{aligned}$$

$$\begin{aligned}-\frac{8}{x^2}+\frac{1}{y}&=0\\ -\frac{x}{y^2}+1&=0\end{aligned}$$

$$\begin{aligned}\frac{-8y+x^2}{x^2y}&=0\\ \frac{-x+y^2}{y^2}&=0\end{aligned}$$

$$\begin{aligned}-8y+x^2&=0\\ -x+y^2&=0\end{aligned}$$

$$\boxed{x=y^2}$$

$$\begin{aligned}-8y+\left(y^2\right)^2&=0\\ -8y+y^4&=0\\ y\left(y^3-8\right)&=0\end{aligned}$$

$y=0$ nije u domeni

$y^3-8=0$

$$\boxed{y=2}$$

$$\boxed{x=4}$$

Stacionarna točka: $(4,2)$

$$\boxed{z_x=-8x^{-2}+y^{-1}} \quad \boxed{z_y=-xy^{-2}+1} \quad \boxed{z=\frac{8}{x}+\frac{x}{y}+y}$$

$$z_{xx}=-8\cdot(-2)x^{-3}=\frac{16}{x^3}$$

$$z_{xy}=-1\cdot y^{-2}=-\frac{1}{y^2}$$

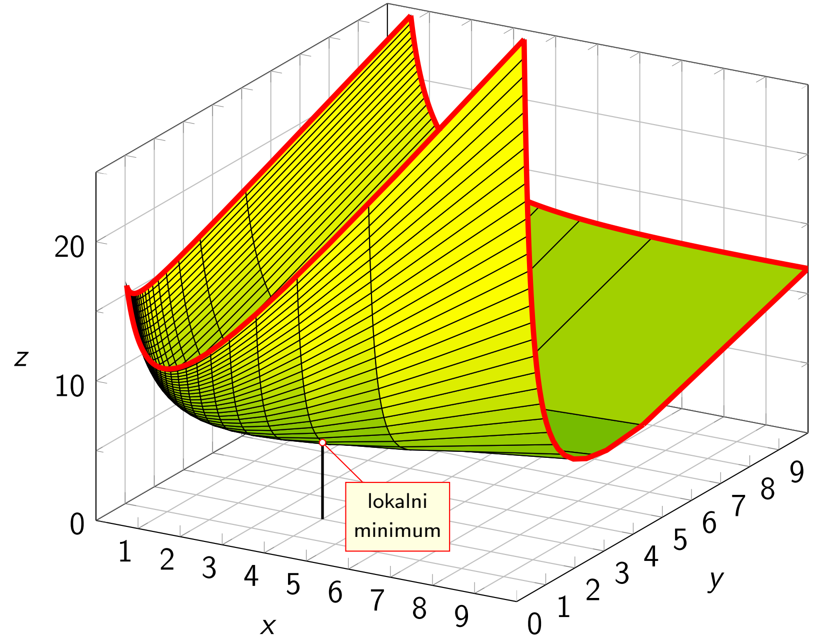
$$z_{yy}=-x\cdot(-2)y^{-3}=\frac{2x}{y^3}$$

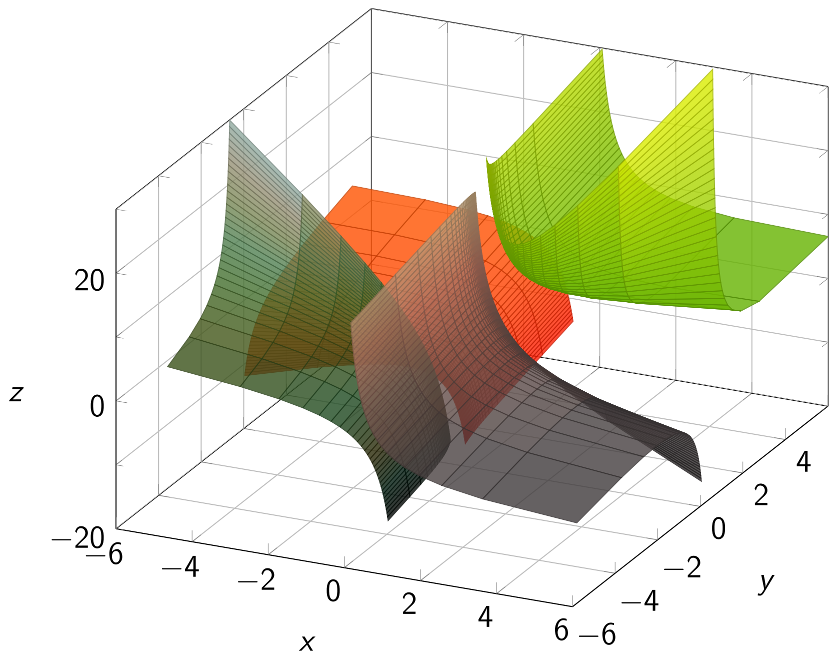
$$H(x,y)=\begin{vmatrix}\frac{16}{x^3} & -\frac{1}{y^2}\\ -\frac{1}{y^2} & \frac{2x}{y^3}\end{vmatrix}$$

$$\overset{x\ \ y}{H(4,2)}=\begin{vmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & 1\end{vmatrix}=\frac{1}{4}-\frac{1}{16}=\frac{3}{16}>0 \longrightarrow \text{točka lokalnog minimuma}$$

>0

$$\overset{x\ \ y}{z(4,2)}=\frac{8}{4}+\frac{4}{2}+2=6$$





Zadatak 9

Odredite ekstreme funkcije $z(x,y) = e^{xy}$ *uz uvjet* $x + y = 4$.

Rješenje

$$x + y = 4 \quad \Longrightarrow \quad \boxed{y = 4 - x}$$

$$z(x, 4-x) = e^{x\cdot(4-x)} = e^{4x-x^2} \longrightarrow f(x) = e^{4x-x^2}$$

$$f'(x) = e^{4x-x^2} \cdot (4x - x^2)' = (4-2x)e^{4x-x^2}$$

$$(4-2x)e^{4x-x^2} = 0$$

$$4 - 2x = 0$$

$$\boxed{x = 2}$$

	$-\infty$	2	$+\infty$
f'	$+$	$-$	
f	$\nearrow$	$\searrow$	

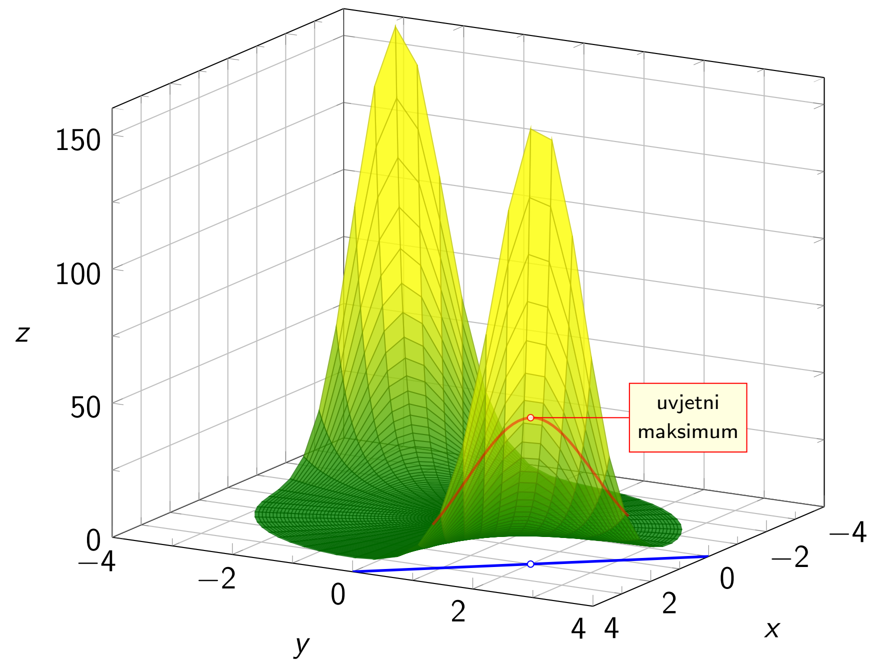
$f(2) = e^4$

globalni maksimum

$$y = 4 - x = 4 - 2 = 2 \qquad \boxed{y = 2}$$

stacionarna točka: $(2,2)$

Funkcija z postiže globalni maksimum uz uvjet $x + y = 4$ u točki $(2,2)$ i taj maksimum je jednak $z(2,2) = e^4$.



Zadatak 10

Odredite ekstreme funkcije $f(x,y) = -x - y$ *uz uvjet* $x^2 + y^2 = 2$.

Rješenje

$$x^2 + y^2 = 2 \longrightarrow x^2 + y^2 - 2 = 0$$

- Lagrangeova funkcija

$$L(x,y,\lambda) = \text{funkcija} + \lambda \cdot \text{uvjet}$$

$$L(x,y,\lambda) = -x - y + \lambda(x^2 + y^2 - 2)$$

- Parcijalne derivacije Lagrangeove funkcije

$$L_x = -1 + 2\lambda x \qquad -1 + 2\lambda x = 0$$

$$L_y = -1 + 2\lambda y \qquad -1 + 2\lambda y = 0$$

$$L_\lambda = x^2 + y^2 - 2 \qquad x^2 + y^2 - 2 = 0$$

$$-1 + 2\lambda x = 0 \longrightarrow 2\lambda x = 1 \longrightarrow \lambda = \frac{1}{2x}$$

$$-1 + 2\lambda y = 0 \longrightarrow 2\lambda y = 1 \longrightarrow \lambda = \frac{1}{2y}$$

$$x^2 + y^2 - 2 = 0$$

$$\Longrightarrow \frac{1}{2x} = \frac{1}{2y} \qquad \boxed{x = y}$$

$$x^2 + x^2 - 2 = 0$$

$$2x^2 = 2$$

$$x^2 = 1$$

$$x_1 = 1, \quad x_2 = -1$$

$$y_1 = 1, \quad y_2 = -1$$

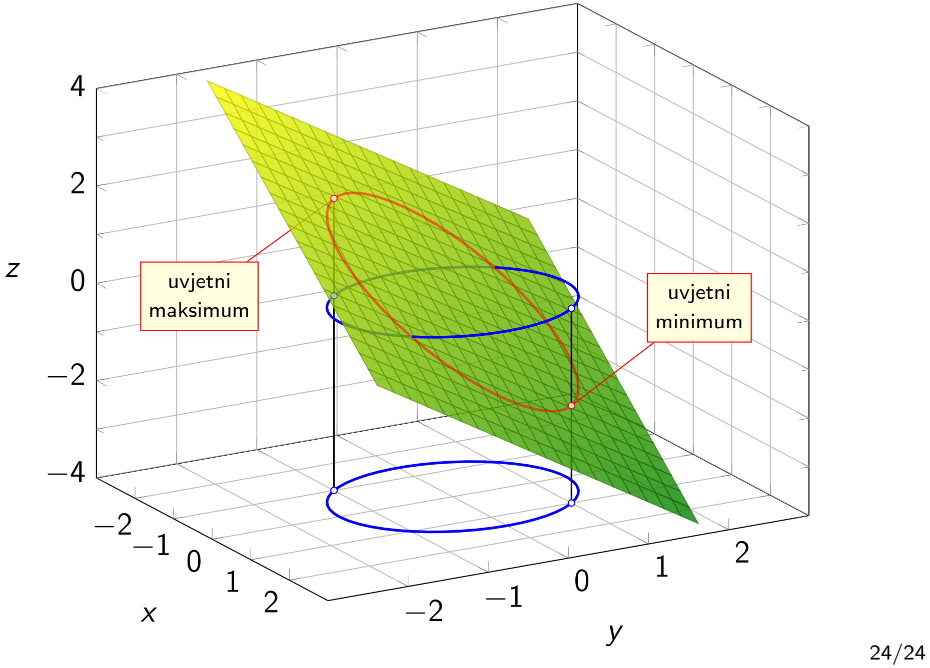
Stacionarne točke: $(1,1)$, $(-1,-1)$

$f(x,y) = -x - y$

- Neprekidna funkcija na kompaktnom skupu (konkretno, omeđenoj krivulji) postiže globalni minimum i globalni maksimum.

$$f(1,1) = -1 - 1 = -2 \longleftarrow \text{minimum}$$

$$f(-1,-1) = -(-1) - (-1) = 2 \longleftarrow \text{maksimum}$$


uvjetni
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