Seminari 5

Matematika za ekonomiste 2

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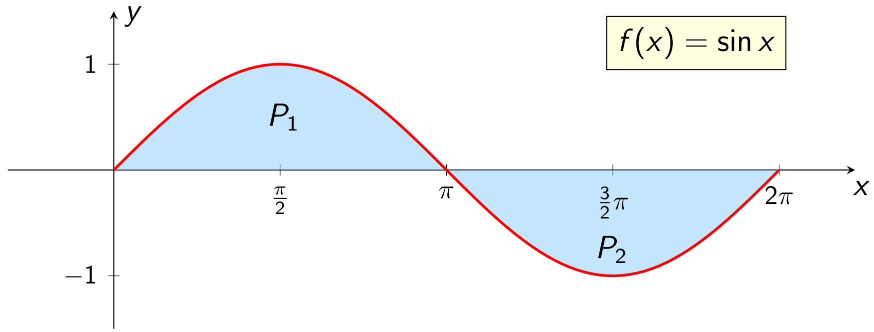
Newton-Leibnizova formula

Teorem

Ako je f neprekidna funkcija na otvorenom intervalu I i F bilo koja primitivna funkcija funkcije f na I, tada za svaki $[a,b] \subseteq I$ vrijedi

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = F(x)\Big|_a^b \qquad F'(x) = f(x),\ x \in [a,b]$$



Vrijednost integrala na segmentu $[0, 2\pi]$

$$\int_0^{2\pi} \sin x\,\mathrm{d}x = -\cos x\Big|_0^{2\pi} = -\cos 2\pi - \big(-\cos 0\big) =$$

$$= -1 - (-1) = -1 + 1 = 0$$

Površina između grafa funkcije i x-osi na segmentu $[0, 2\pi]$

$$P_1 = \int_0^{\pi} \sin x\,\mathrm{d}x = -\cos x\Big|_0^{\pi} = -\cos\pi - \big(-\cos 0\big) =$$

$$= -(-1) - (-1) = 1 + 1 = 2$$

$$P_2 = -\int_{\pi}^{2\pi} \sin x\,\mathrm{d}x = -\big(-\cos x\big)\Big|_{\pi}^{2\pi} = \cos x\Big|_{\pi}^{2\pi} =$$

$$= \cos 2\pi - \cos\pi = 1 - (-1) = 1 + 1 = 2$$

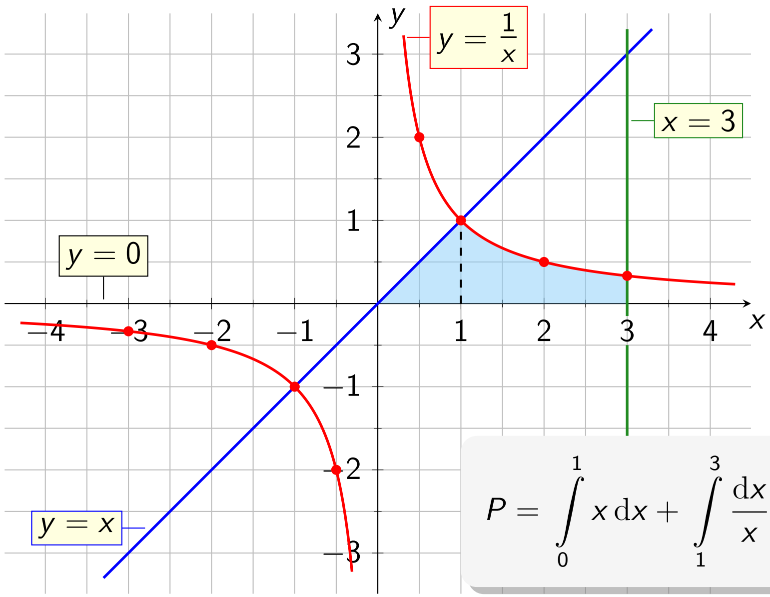
$$P = P_1 + P_2 = 2 + 2 = 4$$

Zadatak 1

Izračunajte površinu lika kojeg omeđuju krivulje

$$y = \frac{1}{x}, \quad y = x, \quad y = 0, \quad x = 3.$$

Rješenje



$$P = \int_0^1 x\,\mathrm{d}x + \int_1^3 \frac{\mathrm{d}x}{x}$$

$$P = \int_0^1 x\mathrm{d}x + \int_1^3 \frac{\mathrm{d}x}{x} = \frac{x^2}{2}\Big|_0^1 + \ln|x|\Big|_1^3 =$$

$$= \left(\frac{1}{2} - 0\right) + (\ln 3 - \ln 1) = \frac{1}{2} + \ln 3$$

Zadatak 2

Izračunajte površinu lika omeđenog grafovima funkcija $f(x) = x^2$ i $g(x) = x + 2$.

$$\boxed{\begin{aligned} ax^2 + bx + c &= 0 \\ x_{1,2} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}}$$

Rješenje

- Presjek pravca i parabole

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$x_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-2)}}{2 \cdot 1}$$

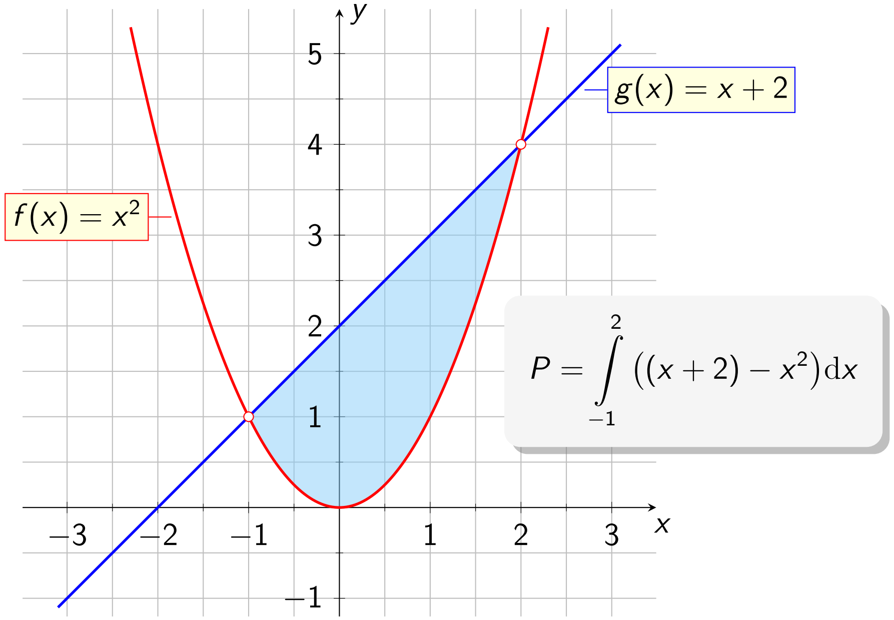
$$x_{1,2} = \frac{1 \pm 3}{2}$$

$$x_1 = 2, \quad x_2 = -1$$

$$y_1 = 4, \quad y_2 = 1$$

točke presjeka

$$\boxed{\begin{aligned} &T_1(2, 4) \\ &T_2(-1, 1) \end{aligned}}$$



$$P = \int_{-1}^{2} \left((x+2) - x^2\right) \mathrm{d}x$$

$$P = \int_{-1}^{2} \left((x+2) - x^2\right) \mathrm{d}x = \int_{-1}^{2} \left(-x^2 + x + 2\right) \mathrm{d}x =$$

$$= \left(-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right)\Bigg|_{-1}^{2} =$$

$$= \left(-\frac{2^3}{3} + \frac{2^2}{2} + 2 \cdot 2\right) - \left(-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2 \cdot (-1)\right) =$$

$$= \left(-\frac{8}{3} + \frac{4}{2} + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) =$$

$$= -\frac{8}{3} + 6 - \frac{1}{3} - \frac{1}{2} + 2 = \frac{9}{2}$$

Zadatak 3

Izračunajte površinu lika kojeg omeđuju krivulje

$$y = \frac{1}{x}, \quad y = 2^{x-1}, \quad y = 4.$$

Rješenje

- Presjek krivulja

$$y = \frac{1}{x} \text{ i } y = 4$$

$$\frac{1}{x} = 4 \;\Big/ \cdot x$$

$$4x = 1$$

$$x = \frac{1}{4}$$

$$\boxed{\left(\frac{1}{4}, 4\right)}$$

- Presjek krivulja

$$y = 2^{x-1} \text{ i } y = 4$$

$$2^{x-1} = 4$$

$$x - 1 = \log_2 4$$

$$x = 2 + 1$$

$$x = 3$$

$$\boxed{(3, 4)}$$

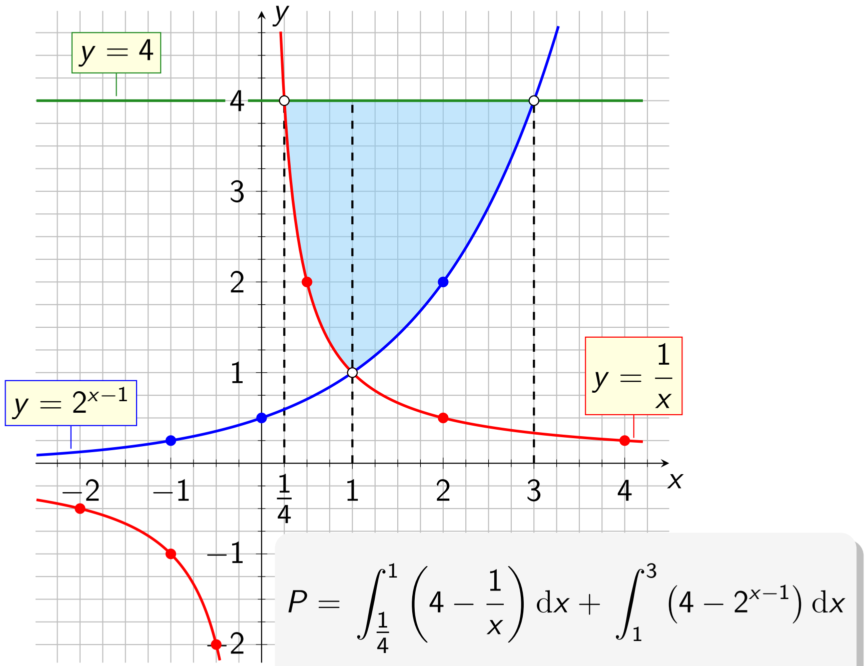
- Presjek krivulja

$$y = 2^{x-1} \text{ i } y = \frac{1}{x}$$

$$2^{x-1} = \frac{1}{x}$$

pogađamo rješenje

$$\boxed{(1, 1)}$$



$$P = \int_{\frac{1}{4}}^{1} \left(4 - \frac{1}{x}\right) \mathrm{d}x + \int_{1}^{3} \left(4 - 2^{x-1}\right) \mathrm{d}x$$

$$\boxed{\int a^x \,\mathrm{d}x = \frac{a^x}{\ln a} + C}$$

$$P = \int_{\frac{1}{4}}^{1} \left(4 - \frac{1}{x}\right) \mathrm{d}x + \int_{1}^{3} \left(4 - 2^{x-1}\right) \mathrm{d}x =$$

$$= \left(4x - \ln|x|\right)\Big|_{\frac{1}{4}}^{1} + \left(4x - \frac{2^{x-1}}{\ln 2}\right)\Big|_{1}^{3} =$$

$$= \left((4 - \ln 1) - \left(1 - \ln\frac{1}{4}\right)\right) + \left(\left(12 - \frac{4}{\ln 2}\right) - \left(4 - \frac{1}{\ln 2}\right)\right) =$$

$$= \left(3 + \ln\frac{1}{4}\right) + \left(8 - \frac{3}{\ln 2}\right) =$$

$$= 11 + \ln\frac{1}{4} - \frac{3}{\ln 2}$$

$P \approx 5.28562$

Zadatak 4

Zadana je funkcija graničnih troškova $T_G = (1+Q)e^{-Q}$.

a) *Odredite za koliko se promijene troškovi ako se proizvodnja s dva proizvoda poveća na pet proizvoda.*

b) *Odredite funkciju troškova ako fiksni troškovi iznose* 100 *novčanih jedinica.*

Rješenje

$$\boxed{\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x}$$

a) $T_G = T'$

$$T' = (1+Q)e^{-Q}$$

$$\boxed{T(Q) = (-Q-2)e^{-Q} + C, \quad C \in \mathbb{R}}$$

$$T = \int (1+Q)e^{-Q}\,\mathrm{d}Q = \int (1+Q)\cdot\left(-e^{-Q}\right)'\mathrm{d}Q =$$

$$= (1+Q)\cdot\left(-e^{-Q}\right) - \int (1+Q)'\cdot\left(-e^{-Q}\right)\mathrm{d}Q =$$

$$= -(1+Q)e^{-Q} - \int 1\cdot\left(-e^{-Q}\right)\mathrm{d}Q =$$

$$= -(1+Q)e^{-Q} + \int e^{-Q}\,\mathrm{d}Q = -(1+Q)e^{-Q} - e^{-Q} + C =$$

$$= (-Q-2)e^{-Q} + C, \quad C \in \mathbb{R}$$

$$\boxed{\int e^x\,\mathrm{d}x = e^x + C}$$

$$T'(Q) = (1+Q)e^{-Q}$$

$$T(Q) = (-Q-2)e^{-Q} + C, \quad C \in \mathbb{R}$$

$$T(5) - T(2) = \int_{2}^{5} T'(Q)\,\mathrm{d}Q = \int_{2}^{5} (1+Q)e^{-Q}\,\mathrm{d}Q =$$

$$= (-Q-2)e^{-Q}\Big|_{2}^{5} = (-5-2)e^{-5} - (-2-2)e^{-2} =$$

$$= 4e^{-2} - 7e^{-5} \approx 0.49418$$

Troškovi se promijene za približno 0.49 novčanih jedinica.

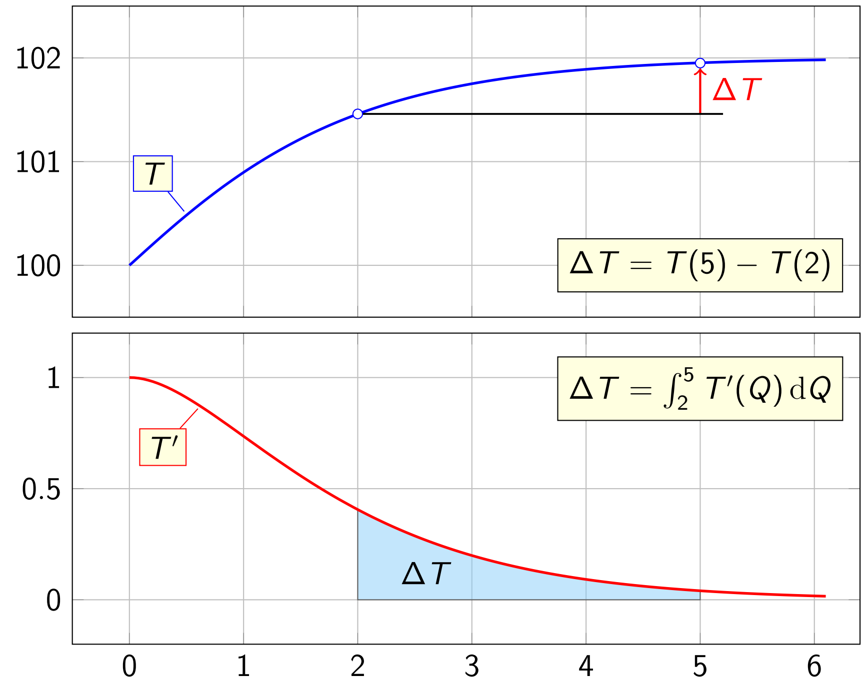
Ova promjena troškova vrijedi za svaki $C \in \mathbb{R}$.

b)

$$T(Q) = (-Q-2)e^{-Q} + C, \quad C \in \mathbb{R}$$

$$T(0) = 100$$
$$(-0-2)\cdot e^{-0} + C = 100$$
$$-2 + C = 100$$
$$C = 102$$

$$T(Q) = (-Q-2)e^{-Q} + 102$$



Zadatak 5

Odredite funkciju potražnje $q(p)$ za koju je $E_{q,p} = -2p$ i $q(0) = 2$.

Rješenje

$$\boxed{\begin{aligned} y &= y(x) \\ y' &= \frac{\mathrm{d}y}{\mathrm{d}x} \end{aligned}}$$

$$E_{q,p} = -2p$$
$$\frac{p}{q}\cdot q' = -2p$$
$$\frac{p}{q}\cdot\frac{\mathrm{d}q}{\mathrm{d}p} = -2p$$
$$\frac{\mathrm{d}q}{q} = -2\,\mathrm{d}p$$
$$\int\frac{\mathrm{d}q}{q} = -2\int\mathrm{d}p$$
$$\ln|q| = -2p + \ln C, \quad C > 0$$

$$\ln|q| - \ln C = -2p$$
$$\ln\frac{|q|}{C} = -2p$$
$$\frac{|q|}{C} = e^{-2p}$$
$$|q| = Ce^{-2p}, \quad C > 0$$
$$q = Ce^{-2p}, \quad C \in \mathbb{R}\setminus\{0\}$$

$$q(0) = 2$$
$$C\cdot e^{-2\cdot 0} = 2$$
$$C = 2$$

$$\boxed{q(p) = 2e^{-2p}}$$

drugi način

$$E_{q,p} = -2p$$
$$\frac{p}{q}\cdot q' = -2p$$
$$\frac{p}{q}\cdot\frac{\mathrm{d}q}{\mathrm{d}p} = -2p$$
$$\frac{\mathrm{d}q}{q} = -2\,\mathrm{d}p$$
$$\int\frac{\mathrm{d}q}{q} = -2\int\mathrm{d}p$$

$$\ln|q| = -2p + C, \quad C \in \mathbb{R}$$
$$|q| = e^{-2p+C}$$
$$q = \pm e^{-2p+C}, \quad C \in \mathbb{R}$$

$$q(0) = 2$$
$$e^{-2\cdot 0 + C} = 2$$
$$e^C = 2$$
$$C = \ln 2$$

$$q = e^{-2p+\ln 2} = e^{-2p}\cdot e^{\ln 2} = 2e^{-2p}$$