Seminari 4

MATEMATIKA ZA EKONOMISTE 2

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Zadatak 1

Riješite neodređeni integral $\int \ln x \,\mathrm{d}x$.

Rješenje

$$\int \ln x \,\mathrm{d}x = \int 1 \cdot \ln x \,\mathrm{d}x = \int (x)' \ln x \,\mathrm{d}x =$$

$$= x \ln x - \int x \cdot (\ln x)' \,\mathrm{d}x = x \ln x - \int x \cdot \frac{1}{x} \,\mathrm{d}x =$$

$$= x \ln x - \int \mathrm{d}x = x \ln x - x + C, \quad C \in \mathbb{R}$$

$$\boxed{\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x}$$

Zadatak 2

Riješite neodređeni integral $\int x^4 \ln 8x \,\mathrm{d}x$.

Rješenje

$$\int x^4 \ln 8x \,\mathrm{d}x = \int \left(\frac{x^5}{5}\right)' \ln 8x \,\mathrm{d}x = \frac{x^5}{5} \ln 8x - \int \frac{x^5}{5} (\ln 8x)' \,\mathrm{d}x =$$

$$= \frac{x^5}{5} \ln 8x - \int \frac{x^5}{5} \cdot \frac{1}{8x} \cdot 8 \,\mathrm{d}x = \frac{x^5}{5} \ln 8x - \frac{1}{5} \int x^4 \,\mathrm{d}x =$$

$$= \frac{x^5}{5} \ln 8x - \frac{1}{5} \cdot \frac{x^5}{5} + C = \frac{x^5}{5} \ln 8x - \frac{1}{25} x^5 + C, \quad C \in \mathbb{R}$$

$$\boxed{\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x}$$

Zadatak 3

Riješite neodređeni integral $\int x \cos 3x \,\mathrm{d}x$.

Rješenje

$$\int x \cos 3x \,\mathrm{d}x = \int x \cdot \left(\frac{1}{3} \sin 3x\right)' \mathrm{d}x =$$

$$= x \cdot \frac{1}{3} \sin 3x - \int (x)' \cdot \frac{1}{3} \sin 3x \,\mathrm{d}x = \frac{x}{3} \sin 3x - \frac{1}{3} \int \sin 3x \,\mathrm{d}x =$$

$$= \frac{x}{3} \sin 3x - \frac{1}{3} \cdot \frac{-1}{3} \cos 3x + C = \frac{x}{3} \sin 3x + \frac{1}{9} \cos 3x + C, \quad C \in \mathbb{R}$$

$$\boxed{\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x}$$

$$\int \cos 3x \,\mathrm{d}x = \left[\begin{array}{c} 3x=t \;/\;' \\ 3\,\mathrm{d}x=\mathrm{d}t \end{array}\right] = \int \cos t \cdot \frac{\mathrm{d}t}{3} =$$

$$= \frac{1}{3}\int \cos t \,\mathrm{d}t = \frac{1}{3}\sin t + C = \frac{1}{3}\sin 3x + C, \quad C \in \mathbb{R}$$

$$\int \sin 3x \,\mathrm{d}x = \left[\begin{array}{c} 3x=t \;/\;' \\ 3\,\mathrm{d}x=\mathrm{d}t \end{array}\right] = \int \sin t \cdot \frac{\mathrm{d}t}{3} =$$

$$= \frac{1}{3}\int \sin t \,\mathrm{d}t = -\frac{1}{3}\cos t + C = -\frac{1}{3}\cos 3x + C, \quad C \in \mathbb{R}$$

Zadatak 4

Riješite neodređeni integral $\int (x^2+x)e^{5x}\,\mathrm{d}x$.

Rješenje

$$\int (x^2+x)e^{5x}\,\mathrm{d}x = \int (x^2+x)\cdot\left(\frac{1}{5}e^{5x}\right)'\mathrm{d}x =$$

$$= (x^2+x)\cdot\frac{1}{5}e^{5x} - \int (x^2+x)'\cdot\frac{1}{5}e^{5x}\,\mathrm{d}x =$$

$$= \left(\frac{1}{5}x^2+\frac{1}{5}x\right)e^{5x} - \frac{1}{5}\int (2x+1)e^{5x}\,\mathrm{d}x =$$

$$= \left(\frac{1}{5}x^2+\frac{1}{5}x\right)e^{5x} - \frac{1}{5}\int (2x+1)\cdot\left(\frac{1}{5}e^{5x}\right)'\mathrm{d}x =$$

$$\boxed{\int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x}$$

$$= \left(\frac{1}{5}x^2+\frac{1}{5}x\right)e^{5x} - \frac{1}{5}\cdot\int (2x+1)\cdot\left(\frac{1}{5}e^{5x}\right)'\mathrm{d}x =$$

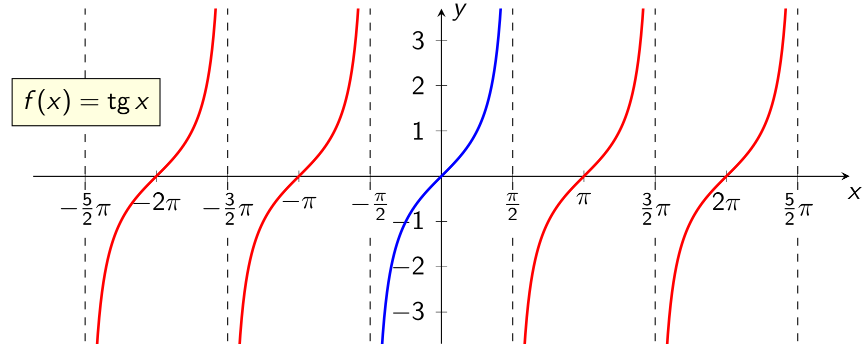
$$= \left(\frac{1}{5}x^2+\frac{1}{5}x\right)e^{5x} - \frac{1}{5}\cdot\left[(2x+1)\cdot\frac{1}{5}e^{5x} - \int (2x+1)'\cdot\frac{1}{5}e^{5x}\,\mathrm{d}x\right] =$$

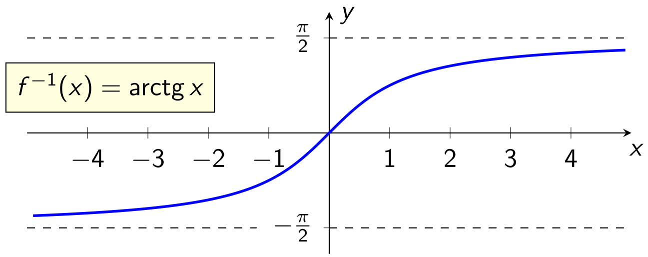
$$= \left(\frac{1}{5}x^2+\frac{1}{5}x\right)e^{5x} - \left(\frac{2}{25}x+\frac{1}{25}\right)e^{5x} + \frac{1}{5}\int 2\cdot\frac{1}{5}e^{5x}\,\mathrm{d}x =$$

$$= \left(\frac{1}{5}x^2+\frac{3}{25}x-\frac{1}{25}\right)e^{5x} + \frac{2}{25}\int e^{5x}\,\mathrm{d}x =$$

$$= \left(\frac{1}{5}x^2+\frac{3}{25}x-\frac{1}{25}\right)e^{5x} + \frac{2}{25}\cdot\frac{1}{5}e^{5x} + C =$$

$$= \left(\frac{1}{5}x^2+\frac{3}{25}x-\frac{3}{125}\right)e^{5x} + C, \quad C \in \mathbb{R}$$





Funkcija

$$f:\left\langle -\frac{\pi}{2},\frac{\pi}{2}\right\rangle \to \mathbb{R},\quad f(x)=\operatorname{tg} x$$

je bijekcija i ima inverznu funkciju

$$f^{-1}:\mathbb{R}\to\left\langle -\frac{\pi}{2},\frac{\pi}{2}\right\rangle,\quad f^{-1}(x)=\operatorname{arctg} x.$$

Derivacija inverzne funkcije jednaka je

$$\left(\operatorname{arctg} x\right)'=\frac{1}{x^2+1}$$

odnosno

$$\int\frac{\mathrm{d}x}{x^2+1}=\operatorname{arctg} x+C,\quad C\in\mathbb{R}.$$

Zadatak 5

Riješite neodređeni integral $\displaystyle\int\frac{x^2}{x^6+1}\,\mathrm{d}x.$

$$\boxed{\begin{aligned}x^3&=t\;/^2\\ x^6&=t^2\end{aligned}}\qquad \boxed{x^2\,\mathrm{d}x=\frac{\mathrm{d}t}{3}}$$

Rješenje

$$\int\frac{x^2}{x^6+1}\,\mathrm{d}x=\left[\begin{array}{c}x^3=t\;/'\\ 3x^2\,\mathrm{d}x=\mathrm{d}t\end{array}\right]=\int\frac{\frac{\mathrm{d}t}{3}}{t^2+1}=\frac{1}{3}\int\frac{\mathrm{d}t}{t^2+1}=$$

$$=\frac{1}{3}\operatorname{arctg} t+C=\frac{1}{3}\operatorname{arctg} x^3+C,\quad C\in\mathbb{R}$$

$$\boxed{\int\frac{\mathrm{d}x}{x^2+1}=\operatorname{arctg} x+C}$$

Zadatak 6

$$\boxed{\int\frac{\mathrm{d}x}{x^2+a^2}=\frac{1}{a}\operatorname{arctg}\frac{x}{a}+C}$$

Riješite neodređeni integral $\displaystyle\int\frac{\mathrm{d}x}{3x^2+5}.$

Rješenje

$$\int\frac{\mathrm{d}x}{3x^2+5}=\int\frac{\mathrm{d}x}{3\left(x^2+\frac{5}{3}\right)}=\frac{1}{3}\int\frac{\mathrm{d}x}{x^2+\sqrt{\frac{5}{3}}^{\,2}}=$$

$$=\frac{1}{3}\cdot\frac{1}{\sqrt{\frac{5}{3}}}\operatorname{arctg}\frac{x}{\sqrt{\frac{5}{3}}}+C=\frac{\sqrt{3}}{3\sqrt{5}}\operatorname{arctg}\frac{\sqrt{3}x}{\sqrt{5}}+C=$$

$$=\frac{\sqrt{15}}{15}\operatorname{arctg}\frac{\sqrt{15}}{5}x+C,\quad C\in\mathbb{R}$$

$$\frac{\sqrt{3}}{3\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{\sqrt{15}}{3\cdot 5}=\frac{\sqrt{15}}{15}\qquad\qquad \frac{\sqrt{3}}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{\sqrt{15}}{5}$$

Zadatak 7

$$\boxed{\int\frac{\mathrm{d}x}{ax+b}=\frac{1}{a}\ln|ax+b|+C}$$

Riješite neodređeni integral $\displaystyle\int\frac{\mathrm{d}x}{x^2-3}.$

Rješenje

$$\frac{1}{x^2-3}=\frac{1}{\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)}=\frac{A}{x-\sqrt{3}}+\frac{B}{x+\sqrt{3}}=$$

$$=\frac{A\left(x+\sqrt{3}\right)+B\left(x-\sqrt{3}\right)}{\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)}$$

$$1=A\left(x+\sqrt{3}\right)+B\left(x-\sqrt{3}\right)$$

$\boxed{x=\sqrt{3}}$

$$1=A\cdot 2\sqrt{3}+B\cdot 0$$

$$A=\frac{1}{2\sqrt{3}}$$

$\boxed{x=-\sqrt{3}}$

$$1=A\cdot 0+B\cdot\left(-2\sqrt{3}\right)$$

$$B=-\frac{1}{2\sqrt{3}}$$

$$\frac{1}{x^2-3}=\frac{\frac{1}{2\sqrt{3}}}{x-\sqrt{3}}+\frac{-\frac{1}{2\sqrt{3}}}{x+\sqrt{3}}$$

$$\int \frac{\mathrm{d}x}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C \qquad \int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{\mathrm{d}x}{x^2-3} = \int \left(\frac{\frac{1}{2\sqrt{3}}}{x-\sqrt{3}} + \frac{-\frac{1}{2\sqrt{3}}}{x+\sqrt{3}} \right) \mathrm{d}x =$$

$$= \frac{1}{2\sqrt{3}} \int \frac{\mathrm{d}x}{x-\sqrt{3}} - \frac{1}{2\sqrt{3}} \int \frac{\mathrm{d}x}{x+\sqrt{3}} =$$

$$= \frac{1}{2\sqrt{3}} \ln\left|x-\sqrt{3}\right| - \frac{1}{2\sqrt{3}} \ln\left|x+\sqrt{3}\right| + C =$$

$$= \frac{1}{2\sqrt{3}} \ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C, \quad C \in \mathbb{R}$$

Zadatak 8

$$\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C$$

Riješite neodređeni integral $\int \frac{\mathrm{d}x}{3x^2+x+4}$.

Rješenje

$$\left(\frac{\frac{1}{3}}{2}\right)^2$$

$$3x^2+x+4 = 3\cdot\left(x^2+\frac{1}{3}x+\frac{4}{3}\right) =$$

$$= 3\cdot\left(\boxed{x^2+\frac{1}{3}x+\frac{1}{36}} - \frac{1}{36} + \frac{4}{3}\right) =$$

$$= 3\cdot\left(\left(x+\frac{1}{6}\right)^2 + \frac{47}{36}\right) =$$

$$= 3\cdot\left(\left(x+\frac{1}{6}\right)^2 + \left(\frac{\sqrt{47}}{6}\right)^2\right)$$

$$\int \frac{\mathrm{d}x}{3x^2+x+4} = \int \frac{\mathrm{d}x}{3\cdot\left(\left(x+\frac{1}{6}\right)^2+\left(\frac{\sqrt{47}}{6}\right)^2\right)} =$$

$$= \frac{1}{3}\int \frac{\mathrm{d}x}{\left(x+\frac{1}{6}\right)^2+\left(\frac{\sqrt{47}}{6}\right)^2} = \left[\begin{array}{c} x+\frac{1}{6} = t \;/\,' \\ \mathrm{d}x = \mathrm{d}t \end{array}\right] =$$

$$= \frac{1}{3}\int \frac{\mathrm{d}t}{t^2+\left(\frac{\sqrt{47}}{6}\right)^2} = \frac{1}{3}\cdot\frac{1}{\frac{\sqrt{47}}{6}}\operatorname{arctg}\frac{t}{\frac{\sqrt{47}}{6}} + C =$$

$$= \frac{2}{\sqrt{47}}\operatorname{arctg}\frac{6t}{\sqrt{47}} + C = \frac{2}{\sqrt{47}}\operatorname{arctg}\frac{6\cdot\left(x+\frac{1}{6}\right)}{\sqrt{47}} + C =$$

$$= \frac{2}{\sqrt{47}}\operatorname{arctg}\frac{6x+1}{\sqrt{47}} + C, \quad C \in \mathbb{R}$$

Zadatak 9

Riješite neodređeni integral $\int \frac{\mathrm{d}x}{x^2+5x-4}$.

Rješenje

$$\left(\frac{5}{2}\right)^2$$

$$x^2+5x-4 = \boxed{x^2+5x+\frac{25}{4}} - \frac{25}{4} - 4 =$$

$$= \left(x+\frac{5}{2}\right)^2 - \frac{41}{4} =$$

$$= \left(x+\frac{5}{2}\right)^2 - \left(\frac{\sqrt{41}}{2}\right)^2$$

$$\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{\mathrm{d}x}{x^2+5x-4} = \int \frac{\mathrm{d}x}{\left(x+\frac{5}{2}\right)^2 - \left(\frac{\sqrt{41}}{2}\right)^2} = \left[\begin{array}{c} x+\frac{5}{2} = t \,/\,' \\ \mathrm{d}x = \mathrm{d}t \end{array}\right] =$$

$$= \int \frac{\mathrm{d}t}{t^2 - \left(\frac{\sqrt{41}}{2}\right)^2} = \frac{1}{2\cdot\frac{\sqrt{41}}{2}} \ln\left|\frac{t-\frac{\sqrt{41}}{2}}{t+\frac{\sqrt{41}}{2}}\right| + C =$$

$$= \frac{1}{\sqrt{41}} \ln\left|\frac{2t-\sqrt{41}}{2t+\sqrt{41}}\right| + C = \frac{1}{\sqrt{41}} \ln\left|\frac{2\cdot\left(x+\frac{5}{2}\right)-\sqrt{41}}{2\cdot\left(x+\frac{5}{2}\right)+\sqrt{41}}\right| + C =$$

$$= \frac{1}{\sqrt{41}} \ln\left|\frac{2x+5-\sqrt{41}}{2x+5+\sqrt{41}}\right| + C, \quad C\in\mathbb{R}$$

Zadatak 10

Riješite neodređeni integral $\int \frac{5x+3}{x^2+5x-4}\,\mathrm{d}x$.

$$\boxed{\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C}$$

Rješenje

$$\int \frac{5x+3}{x^2+5x-4}\,\mathrm{d}x = \int \frac{\frac{5}{2}\cdot(2x+5)-\frac{19}{2}}{x^2+5x-4}\,\mathrm{d}x =$$

$$= \frac{5}{2}\int \frac{2x+5}{x^2+5x-4}\,\mathrm{d}x - \frac{19}{2}\boxed{\int \frac{\mathrm{d}x}{x^2+5x-4}} =$$

prethodni zadatak

$$= \frac{5}{2}\ln\left|x^2+5x-4\right| - \frac{19}{2}\cdot\frac{1}{\sqrt{41}}\ln\left|\frac{2x+5-\sqrt{41}}{2x+5+\sqrt{41}}\right| + C =$$

$$= \frac{5}{2}\ln\left|x^2+5x-4\right| - \frac{19}{2\sqrt{41}}\ln\left|\frac{2x+5-\sqrt{41}}{2x+5+\sqrt{41}}\right| + C, \quad C\in\mathbb{R}$$