Sustavi linearnih jednadžbi. Kronecker-Capellijev teorem

MATEMATIKA ZA EKONOMISTE 1

Damir Horvat

FOI, Varaždin

Zadatak 1

Odredite rang matrice

$$A=\begin{array}{cc}\begin{bmatrix}-4 & -3 & -4 & -1 & 0\\ 1 & 1 & 1 & 1 & 3\\ 14 & 11 & 14 & 5 & 2\\ 11 & 9 & 11 & 5 & 4\end{bmatrix} & \begin{array}{l}\leftarrow 1.\\ \leftarrow 2.\\ \leftarrow 3.\\ \leftarrow 4.\end{array}\\ \begin{array}{ccccc}\uparrow & \uparrow & \uparrow & \uparrow & \uparrow\\ 1. & 2. & 3. & 4. & 5.\end{array} & \end{array}$$

subdeterminanta reda 2

$$\begin{array}{cc}\begin{vmatrix}1 & 3\\ 11 & 4\end{vmatrix} & \begin{array}{l}\leftarrow 2.\\ \leftarrow 4.\end{array}\\ \begin{array}{cc}\uparrow & \uparrow\\ 1. & 5.\end{array} & \end{array}$$

subdeterminanta reda 3

$$\begin{array}{cc}\begin{vmatrix}-4 & -4 & -1\\ 1 & 1 & 1\\ 14 & 14 & 5\end{vmatrix} & \begin{array}{l}\leftarrow 1.\\ \leftarrow 2.\\ \leftarrow 3.\end{array}\\ \begin{array}{ccc}\uparrow & \uparrow & \uparrow\\ 1. & 3. & 4.\end{array} & \end{array}$$

subdeterminanta reda 4

$$\begin{array}{cc}\begin{vmatrix}-4 & -3 & -4 & 0\\ 1 & 1 & 1 & 3\\ 14 & 11 & 14 & 2\\ 11 & 9 & 11 & 4\end{vmatrix} & \begin{array}{l}\leftarrow 1.\\ \leftarrow 2.\\ \leftarrow 3.\\ \leftarrow 4.\end{array}\\ \begin{array}{cccc}\uparrow & \uparrow & \uparrow & \uparrow\\ 1. & 2. & 3. & 5.\end{array} & \end{array}$$

Rješenje

$$\begin{bmatrix}-4 & -3 & -4 & -1 & 0\\ 1 & 1 & 1 & 1 & 3\\ 14 & 11 & 14 & 5 & 2\\ 11 & 9 & 11 & 5 & 4\end{bmatrix}\sim\begin{bmatrix}1 & 1 & 1 & 1 & 3\\ -4 & -3 & -4 & -1 & 0\\ 14 & 11 & 14 & 5 & 2\\ 11 & 9 & 11 & 5 & 4\end{bmatrix}/\cdot 4\,/\cdot(-14)\,/\cdot(-11)\;\sim$$

$$\sim\begin{bmatrix}1 & 1 & 1 & 1 & 3\\ 0 & 1 & 0 & 3 & 12\\ 0 & -3 & 0 & -9 & -40\\ 0 & -2 & 0 & -6 & -29\end{bmatrix}/\cdot 3\,/\cdot 2\;\sim\begin{bmatrix}1 & 1 & 1 & 1 & 3\\ 0 & 1 & 0 & 3 & 12\\ 0 & 0 & 0 & 0 & -4\\ 0 & 0 & 0 & 0 & -5\end{bmatrix}\sim$$

$$\sim\begin{bmatrix}1 & 1 & 3 & 1 & 1\\ 0 & 1 & 12 & 3 & 0\\ 0 & 0 & -4 & 0 & 0\\ 0 & 0 & -5 & 0 & 0\end{bmatrix}/\cdot\frac{-5}{4}\;\sim\begin{bmatrix}1 & 1 & 3 & 1 & 1\\ 0 & 1 & 12 & 3 & 0\\ 0 & 0 & -4 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\boxed{r(A)=3}$$

$$\begin{vmatrix}1 & 1 & 3\\ 0 & 1 & 12\\ 0 & 0 & -4\end{vmatrix}\neq 0$$

Broj parametara u rješivom sustavu

broj parametara = **broj nepoznanica** − **broj nezavisnih jednadžbi**

broj nezavisnih jednadžbi = **rang proširene matrice sustava** = **rang matrice sustava**

ako je sustav konzistentan

Zadatak 2

Zadan je sustav linearnih jednadžbi

$$
\begin{aligned}
2x_1 + 3x_2 + 2x_3 + 6x_4 &= 1 \\
-2x_1 + 3x_2 - 6x_3 + 12x_4 &= -19. \\
2x_1 + 6x_2 + 15x_4 &= -8
\end{aligned}
$$

a) *Pomoću Kronecker-Capellijevog teorema ispitajte koliko rješenja ima zadani sustav.*

b) *Riješite zadani sustav Gaussovim postupkom.*

Rješenje

a)

$$
A = \begin{bmatrix} 2 & 3 & 2 & 6 \\ -2 & 3 & -6 & 12 \\ 2 & 6 & 0 & 15 \end{bmatrix}
\qquad
A_p = \left[\begin{array}{cccc:c} 2 & 3 & 2 & 6 & 1 \\ -2 & 3 & -6 & 12 & -19 \\ 2 & 6 & 0 & 15 & -8 \end{array}\right]
$$

$$
\left[\begin{array}{cccc:c} 2 & 3 & 2 & 6 & 1 \\ -2 & 3 & -6 & 12 & -19 \\ 2 & 6 & 0 & 15 & -8 \end{array}\right] /\cdot 1 \,/\cdot(-1)
\sim
\left[\begin{array}{cccc:c} 2 & 3 & 2 & 6 & 1 \\ 0 & 6 & -4 & 18 & -18 \\ 0 & 3 & -2 & 9 & -9 \end{array}\right] \sim
$$

$$
\sim \left[\begin{array}{cccc:c} 2 & 3 & 2 & 6 & 1 \\ 0 & 3 & -2 & 9 & -9 \\ 0 & 6 & -4 & 18 & -18 \end{array}\right] /\cdot(-2)
\sim
\left[\begin{array}{cccc:c} 2 & 3 & 2 & 6 & 1 \\ 0 & 3 & -2 & 9 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

$$\boxed{r(A) = 2} \qquad \boxed{r(A_p) = 2}$$

$$r(A) = r(A_p) = 2 < \text{broj nepoznanica}$$

zadani sustav ima rješenje

zadani sustav ima beskonačno mnogo rješenja

b) broj parametara u općem rješenju

$$
\begin{aligned}
&\text{broj parametara} = \text{broj nepoznanica} - r(A) \\
&\text{broj parametara} = 4 - 2 \\
&\text{broj parametara} = 2
\end{aligned}
$$

$$
\begin{array}{cccc|cl}
x_1 & \boxed{x_2} & x_3 & \boxed{x_4} & & \\
\hline
2 & 3 & 2 & 6 & 1 & \\
-2 & 3 & -6 & 12 & -19 & \\
2 & 6 & 0 & 15 & -8 & /\cdot 1 \,/\cdot(-1) \\
\hline
0 & -3 & 2 & -9 & 9 & /\cdot 3 \\
0 & 9 & -6 & 27 & -27 & \\
2 & 6 & 0 & 15 & -8 & \\
\hline
0 & -3 & 2 & -9 & 9 & \\
0 & 0 & 0 & 0 & 0 & \\
2 & 6 & 0 & 15 & -8 & \\
\hline
0 & -3 & 2 & -9 & 9 & \\
2 & 6 & 0 & 15 & -8 &
\end{array}
$$

$$
\left.\begin{aligned}
-3x_2 + 2x_3 - 9x_4 &= 9 \\
2x_1 + 6x_2 + 15x_4 &= -8
\end{aligned}\right\}
$$

$$
\begin{aligned}
-3x_2 + 2x_3 - 9x_4 &= 9 \\
2x_3 &= 9 + 3x_2 + 9x_4
\end{aligned}
$$

$$\boxed{x_3 = \frac{9}{2} + \frac{3}{2}x_2 + \frac{9}{2}x_4}$$

$$
\begin{aligned}
2x_1 + 6x_2 + 15x_4 &= -8 \\
2x_1 &= -8 - 6x_2 - 15x_4
\end{aligned}
$$

$$\boxed{x_1 = -4 - 3x_2 - \frac{15}{2}x_4}$$

Opće rješenje sustava

$$
\begin{aligned}
x_1 &= -4 - 3u - \frac{15}{2}v \\
x_2 &= u \\
x_3 &= \frac{9}{2} + \frac{3}{2}u + \frac{9}{2}v \\
x_4 &= v
\end{aligned}
\qquad u, v \in \mathbb{R}
$$

$$\boxed{\begin{aligned} x_1 &= -4 - 3u - \tfrac{15}{2}v \\ x_2 &= u \\ x_3 &= \tfrac{9}{2} + \tfrac{3}{2}u + \tfrac{9}{2}v \\ x_4 &= v \end{aligned}}$$

Neka posebna rješenja

- bazično rješenje: $u = 0,\ v = 0$

$$x_1 = -4,\ x_2 = 0,\ x_3 = \frac{9}{2},\ x_4 = 0$$

- $u = 0,\ v = 1$

$$x_1 = -\frac{23}{2},\ x_2 = 0,\ x_3 = 9,\ x_4 = 1$$

- $u = \sqrt{2},\ v = \pi$

$$x_1 = -4 - 3\sqrt{2} - \frac{15}{2}\pi,\quad x_2 = \sqrt{2},\quad x_3 = \frac{9}{2} + \frac{3}{2}\sqrt{2} + \frac{9}{2}\pi,\quad x_4 = \pi$$

Zadatak 3

Pomoću Kronecker-Capellijevog teorema ispitajte koliko rješenja ima sustav linearnih jednadžbi

$$\begin{aligned} x_1 - 4x_2 + 5x_3 &= 6 \\ -3x_2 + 2x_3 &= -12. \\ 2x_1 + 7x_2 &= 35 \end{aligned}$$

Rješenje

$$A = \begin{bmatrix} 1 & -4 & 5 \\ 0 & -3 & 2 \\ 2 & 7 & 0 \end{bmatrix} \qquad A_p = \left[\begin{array}{ccc|c} 1 & -4 & 5 & 6 \\ 0 & -3 & 2 & -12 \\ 2 & 7 & 0 & 35 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -4 & 5 & 6 \\ 0 & -3 & 2 & -12 \\ 2 & 7 & 0 & 35 \end{array}\right] / \cdot (-2) \;+ \sim \left[\begin{array}{ccc|c} 1 & -4 & 5 & 6 \\ 0 & -3 & 2 & -12 \\ 0 & 15 & -10 & 23 \end{array}\right] / \cdot 5 \;+ \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & 5 & 6 \\ 0 & -3 & 2 & -12 \\ 0 & 0 & 0 & -37 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -4 & 6 & 5 \\ 0 & -3 & -12 & 2 \\ 0 & 0 & -37 & 0 \end{array}\right]$$

$$\boxed{r(A) = 2} \qquad \boxed{r(A_p) = 3}$$

$r(A) \neq r(A_p) \longrightarrow$ zadani sustav je kontradiktoran

Zadatak 4

Zadan je homogeni sustav linearnih jednadžbi

$$\begin{aligned} 6x_1 - 4x_2 + x_3 &= 0 \\ -x_1 + x_2 + 4x_3 &= 0. \\ 4x_1 - 2x_2 + ax_3 &= 0 \end{aligned}$$

a) *Odredite sve vrijednosti parametra $a \in \mathbb{R}$ za koje sustav ima i netrivijalnih rješenja.*

b) *Za sve pronađene vrijednosti parametra $a \in \mathbb{R}$ iz a) dijela zadatka riješite pripadni sustav jednadžbi.*

Rješenje

a) Roucheov teorem

$$\begin{cases} 6x_1 - 4x_2 + x_3 = 0 \\ -x_1 + x_2 + 4x_3 = 0 \\ 4x_1 - 2x_2 + ax_3 = 0 \end{cases}$$

$$\left|\begin{array}{ccc|cc} 6 & -4 & 1 & 6 & -4 \\ -1 & 1 & 4 & -1 & 1 \\ 4 & -2 & a & 4 & -2 \end{array}\right. =$$

$$= 6 \cdot 1 \cdot a + (-4) \cdot 4 \cdot 4 + 1 \cdot (-1) \cdot (-2) -$$

$$- 4 \cdot 1 \cdot 1 - (-2) \cdot 4 \cdot 6 - a \cdot (-1) \cdot (-4) =$$

$$= 6a - 64 + 2 - 4 + 48 - 4a = 2a - 18$$

$$2a - 18 = 0$$

$$\boxed{a = 9}$$

- Za $a = 9$ pripadni homogeni sustav ima i netrivijalnih rješenja.
- Za $a \in \mathbb{R} \setminus \{9\}$ pripadni homogeni sustav ima samo trivijalno rješenje.

b) $\boxed{a = 9}$

$$\begin{cases} 6x_1 - 4x_2 + x_3 = 0 \\ -x_1 + x_2 + 4x_3 = 0 \\ 4x_1 - 2x_2 + ax_3 = 0 \end{cases}$$

$$\begin{array}{ccc|cl} x_1 & x_2 & \boxed{x_3} & & \\ \hline 6 & -4 & 1 & 0 & + \\ -1 & 1 & 4 & 0 & /\cdot 6 \,/\cdot 4 \\ 4 & -2 & 9 & 0 & + \\ \hline 0 & 2 & 25 & 0 & \\ -1 & 1 & 4 & 0 & \\ 0 & 2 & 25 & 0 & \\ \hline 0 & 2 & 25 & 0 & /\cdot \frac{-1}{2} \\ -1 & 1 & 4 & 0 & + \\ \hline 0 & 2 & 25 & 0 & \\ -1 & 0 & -\frac{17}{2} & 0 & \end{array}$$

$$\left.\begin{array}{r} 2x_2 + 25x_3 = 0 \\ -x_1 - \frac{17}{2}x_3 = 0 \end{array}\right\}$$

$$\boxed{x_2 = -\frac{25}{2}x_3} \qquad \boxed{x_1 = -\frac{17}{2}x_3}$$

Opće rješenje sustava

$$\left(-\frac{17}{2}t,\ -\frac{25}{2}t,\ t\right), \quad t \in \mathbb{R}$$