Sustavi linearnih jednadžbi. Gaussov postupak

Matematika za ekonomiste 1

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Sustavi linearnih jednadžbi

Zadatak 1

Riješite sustav linearnih jednadžbi

$$\begin{aligned} 2x - y + 2z &= 1 \\ x - 3y + z &= 2\,. \\ 4x - 2y + 3z &= -4 \end{aligned}$$

Rješenje pomoću inverzne matrice

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & -3 & 1 \\ 4 & -2 & 3 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

$$X = \frac{1}{5}\begin{bmatrix} -7 & -1 & 5 \\ 1 & -2 & 0 \\ 10 & 0 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$$

$$X = \frac{1}{5}\begin{bmatrix} -29 \\ -3 \\ 30 \end{bmatrix}$$

DZ

$$A^{-1} = \frac{1}{5}\begin{bmatrix} -7 & -1 & 5 \\ 1 & -2 & 0 \\ 10 & 0 & -5 \end{bmatrix}$$

$$X = \begin{bmatrix} -\frac{29}{5} \\ -\frac{3}{5} \\ 6 \end{bmatrix} \qquad \begin{aligned} x &= -\frac{29}{5} \\ y &= -\frac{3}{5} \\ z &= 6 \end{aligned}$$

Cramerovo pravilo

$$D = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -3 & 1 \\ 4 & -2 & 3 \end{vmatrix} = 5 \qquad D_1 = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -3 & 1 \\ -4 & -2 & 3 \end{vmatrix} = -29$$

$$D_2 = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 4 & -4 & 3 \end{vmatrix} = -3 \qquad D_3 = \begin{vmatrix} 2 & -1 & 1 \\ 1 & -3 & 2 \\ 4 & -2 & -4 \end{vmatrix} = 30$$

$$x = \frac{D_1}{D} = \frac{-29}{5} \qquad y = \frac{D_2}{D} = \frac{-3}{5}$$

$$z = \frac{D_3}{D} = \frac{30}{5} = 6$$

$$\begin{aligned} 2x - y + 2z &= 1 \\ x - 3y + z &= 2 \\ 4x - 2y + 3z &= -4 \end{aligned}$$

Gaussov postupak

x	y	z	
2	-1	2	1
(1)	-3	1	2
4	-2	3	-4
0	5	0	-3
1	-3	1	2
0	10	(-1)	-12
0	5	0	-3
1	7	0	-10
0	10	-1	-12

$/\cdot(-2)$ $/\cdot(-4)$ $/\cdot 1$ $/\cdot\frac{1}{5}$

$$2x - y + 2z = 1$$
$$x - 3y + z = 2$$
$$4x - 2y + 3z = -4$$

x	y	z	
0	(1)	0	$-\frac{3}{5}$
1	7	0	-10
0	10	-1	-12
0	1	0	$-\frac{3}{5}$
1	0	0	$-\frac{29}{5}$
0	0	-1	-6

$/\cdot(-7)$ $/\cdot(-10)$

$$y = -\frac{3}{5}$$
$$x = -\frac{29}{5}$$
$$-z = -6 \longrightarrow z = 6$$

Zadatak 2

Zadan je sustav linearnih jednadžbi

$$x_1 - 3x_2 + 5x_3 = 1$$
$$4x_1 + 28x_2 - 28x_3 = 0\,.$$
$$4x_1 + 8x_2 - 4x_3 = 2$$

a) *Gaussovim postupkom riješite zadani sustav tako da varijabla x_1 bude parametar.*

b) *Pronađite sva bazična rješenja.*

c) *Odredite ono rješenje sustava čija je suma komponenata jednaka 0.*

Rješenje

a)

parametar

suvišna jednadžba

x_1	x_2	x_3	
1	-3	5	1
4	28	-28	0
4	8	-4	2
1	-3	5	1
1	7	-7	0
1	2	(-1)	$\frac{1}{2}$
6	(7)	0	$\frac{7}{2}$
-6	-7	0	$-\frac{7}{2}$
1	2	-1	$\frac{1}{2}$

$/:4$ $/:4$ $/\cdot(-7)$ $/\cdot 5$ $/\cdot 1$ $/\cdot\frac{-2}{7}$

x_1	x_2	x_3	
6	7	0	$\frac{7}{2}$
0	0	0	0
$-\frac{5}{7}$	0	-1	$-\frac{1}{2}$
6	7	0	$\frac{7}{2}$
$-\frac{5}{7}$	0	-1	$-\frac{1}{2}$

$$\left.\begin{aligned} 6x_1 + 7x_2 &= \tfrac{7}{2} \\ -\tfrac{5}{7}x_1 - x_3 &= -\tfrac{1}{2} \end{aligned}\right\}$$

$$x_1 - 3x_2 + 5x_3 = 1$$
$$4x_1 + 28x_2 - 28x_3 = 0$$
$$4x_1 + 8x_2 - 4x_3 = 2$$

$$6x_1 + 7x_2 = \tfrac{7}{2} \longrightarrow 7x_2 = -6x_1 + \tfrac{7}{2} \,/\cdot\tfrac{1}{7} \longrightarrow \boxed{x_2 = -\tfrac{6}{7}x_1 + \tfrac{1}{2}}$$
$$-\tfrac{5}{7}x_1 - x_3 = -\tfrac{1}{2} \longrightarrow -x_3 = \tfrac{5}{7}x_1 - \tfrac{1}{2} \,/\cdot(-1) \longrightarrow \boxed{x_3 = -\tfrac{5}{7}x_1 + \tfrac{1}{2}}$$

Opće rješenje sustava

1. način zapisivanja

$$x_2 = -\tfrac{6}{7}x_1 + \tfrac{1}{2}$$
$$x_3 = -\tfrac{5}{7}x_1 + \tfrac{1}{2}$$
$$x_1 \in \mathbb{R}$$

2. način zapisivanja

$$x_1 = p$$
$$x_2 = -\tfrac{6}{7}p + \tfrac{1}{2}$$
$$x_3 = -\tfrac{5}{7}p + \tfrac{1}{2}$$
$$p \in \mathbb{R}$$

3. način zapisivanja

$$\left(p,\ -\tfrac{6}{7}p + \tfrac{1}{2},\ -\tfrac{5}{7}p + \tfrac{1}{2}\right)$$
$$p \in \mathbb{R}$$

$$x_1 = p$$
$$x_2 = -\frac{6}{7}p + \frac{1}{2}$$
$$x_3 = -\frac{5}{7}p + \frac{1}{2}$$

b) Bazična rješenja

$x_1 = 0$

$p = 0$

$\left(0, \frac{1}{2}, \frac{1}{2}\right)$

$x_2 = 0$

$-\frac{6}{7}p + \frac{1}{2} = 0 \,/\cdot 14$

$-12p + 7 = 0$

$p = \frac{7}{12}$

$\left(\frac{7}{12}, 0, \frac{1}{12}\right)$

$x_3 = 0$

$-\frac{5}{7}p + \frac{1}{2} = 0 \,/\cdot 14$

$-10p + 7 = 0$

$p = \frac{7}{10}$

$\left(\frac{7}{10}, -\frac{1}{10}, 0\right)$

c)

$$x_1 + x_2 + x_3 = 0$$

$$p + \left(-\frac{6}{7}p + \frac{1}{2}\right) + \left(-\frac{5}{7}p + \frac{1}{2}\right) = 0$$

$$-\frac{4}{7}p + 1 = 0$$

$$p = \frac{7}{4}$$

$$x_1 = p$$
$$x_2 = -\frac{6}{7}p + \frac{1}{2}$$
$$x_3 = -\frac{5}{7}p + \frac{1}{2}$$

Traženo rješenje sustava $\left(\frac{7}{4}, -1, -\frac{3}{4}\right)$

$$x_1 = \frac{7}{4}, \quad x_2 = -1, \quad x_3 = -\frac{3}{4}$$

Broj parametara u rješivom sustavu

U prethodnom zadatku
broj parametara $= 3 - 2 = 1$

nakon provedenog Gaussovog postupka znamo taj podatak

broj parametara = **broj nepoznanica** − **broj nezavisnih jednadžbi**

taj podatak je povezan s rangom matrice sustava

Kronecker-Capellijev teorem

Strpite se.
Predivni Kronecker-Capellijev teorem i rang matrice radit ćemo na sljedećim seminarima.

Zadatak 3

Riješite sustav linearnih jednadžbi

$$\begin{aligned} x_1 - 4x_2 + 5x_3 &= 6 \\ -3x_2 + 2x_3 &= -12 \\ 2x_1 + 7x_2 &= 35 \end{aligned}$$

Rješenje

x_1	x_2	x_3	
①	-4	5	6
0	-3	2	-12
2	7	0	35
1	-4	5	6
0	-3	②	-12
0	15	-10	23
1	$\frac{7}{2}$	0	36
0	-3	2	-12
0	0	0	-37

$/\cdot(-2)$, $+$, $+$, $/\cdot\frac{-5}{2}$, $/\cdot 5$, $+$

$0 = -37$

sustav je kontradiktoran

2	3	1	1	0	0	
①	-3	2	0	1	0	$/\cdot(-2)$ $/\cdot 2$
-2	6	-1	0	0	1	
0	9	-3	1	-2	0	$/:9$
1	-3	2	0	1	0	
0	0	3	0	2	1	
0	①	$-\frac{1}{3}$	$\frac{1}{9}$	$-\frac{2}{9}$	0	$/\cdot 3$
1	-3	2	0	1	0	
0	0	3	0	2	1	
0	1	$-\frac{1}{3}$	$\frac{1}{9}$	$-\frac{2}{9}$	0	
1	0	1	$\frac{1}{3}$	$\frac{1}{3}$	0	
0	0	3	0	2	1	$/:3$

$+$, $+$, $+$, $+$

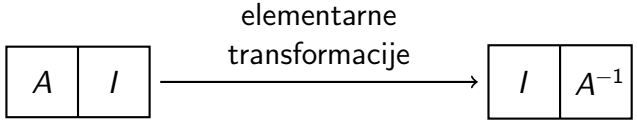
$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -3 & 2 \\ -2 & 6 & -1 \end{bmatrix}$$

Zadatak 4

Gaussovim postupkom pronađite inverznu matricu matrice

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -3 & 2 \\ -2 & 6 & -1 \end{bmatrix}.$$

Rješenje



0	1	$-\frac{1}{3}$	$\frac{1}{9}$	$-\frac{2}{9}$	0	
1	0	1	$\frac{1}{3}$	$\frac{1}{3}$	0	
0	0	①	0	$\frac{2}{3}$	$\frac{1}{3}$	$/\cdot(-1)$ $/\cdot\frac{1}{3}$
0	1	0	$\frac{1}{9}$	0	$\frac{1}{9}$	2.
1	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	1.
0	0	1	0	$\frac{2}{3}$	$\frac{1}{3}$	3.
1	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	
0	1	0	$\frac{1}{9}$	0	$\frac{1}{9}$	
0	0	1	0	$\frac{2}{3}$	$\frac{1}{3}$	

$+$, $+$, $+$

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{9} & 0 & \frac{1}{9} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$