Međusektorska analiza

Matematika za ekonomiste 1

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Zadatak 1

Zadana je input-output tablica dvosektorskog modela ekonomije.

	X_{ij}		Y_i	X_i
	1	2		
1	220	540	340	1100
2	330	202.5	817.5	1350

a) *Odredite matricu tehničkih koeficijenata.*

b) *Odredite matricu tehnologije.*

c) *Odredite matricu multiplikator.*

d) *Odredite novu ukupnu proizvodnju ako je nova finalna potražnja* $Y = (388, 888)$.

e) *Napravite novu međusektorsku tablicu na temelju nove finalne potražnje.*

Međusektorska analiza (input – output analiza)

- Matrica tehničkih koeficijenata

$$A = [a_{ij}], \qquad a_{ij} = \frac{X_{ij}}{X_j}$$

- Matrica tehnologije

$$T = I - A$$

- Matrica multiplikator

$$M = T^{-1} = (I - A)^{-1}$$

- Strukturni oblik input-output modela

$$(I - A)X = Y$$

vektor ukupne proizvodnje

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

vektor finalne potražnje

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$$

Rješenje

a) Matrica tehničkih koeficijenata

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{3}{10} & \frac{3}{20} \end{bmatrix}$$

$$a_{11} = \frac{X_{11}}{X_1} = \frac{220}{1100} = \frac{1}{5} \qquad a_{12} = \frac{X_{12}}{X_2} = \frac{540}{1350} = \frac{2}{5}$$

$$a_{21} = \frac{X_{21}}{X_1} = \frac{330}{1100} = \frac{3}{10} \qquad a_{22} = \frac{X_{22}}{X_2} = \frac{202.5}{1350} = \frac{3}{20}$$

$a_{ij} = \frac{X_{ij}}{X_j}$

$$\frac{202.5}{1350} = \frac{202.5 \cdot 10}{1350 \cdot 10} = \frac{2025}{13\,500} = \frac{3}{20}$$

b) Matrica tehnologije

$$T = I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{3}{10} & \frac{3}{20} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{3}{10} & \frac{17}{20} \end{bmatrix}$$

$$T = \frac{1}{20} \begin{bmatrix} 16 & -8 \\ -6 & 17 \end{bmatrix}$$

c) Matrica multiplikator

$$T = \frac{1}{20} \begin{bmatrix} 16 & -8 \\ -6 & 17 \end{bmatrix}$$

$$M = (I - A)^{-1} = T^{-1}$$

$$M = \left(\frac{1}{20}\right)^{-1} \begin{bmatrix} 16 & -8 \\ -6 & 17 \end{bmatrix}^{-1}$$

$$M = 20 \cdot \frac{1}{16 \cdot 17 - (-6) \cdot (-8)} \begin{bmatrix} 17 & 8 \\ 6 & 16 \end{bmatrix}$$

$$M = 20 \cdot \frac{1}{224} \begin{bmatrix} 17 & 8 \\ 6 & 16 \end{bmatrix} \qquad M = \frac{5}{56} \begin{bmatrix} 17 & 8 \\ 6 & 16 \end{bmatrix}$$

d)

$$(I - A)X = Y$$

$$X = (I - A)^{-1} Y$$

$$X = \frac{5}{56} \begin{bmatrix} 17 & 8 \\ 6 & 16 \end{bmatrix} \begin{bmatrix} 388 \\ 888 \end{bmatrix}$$

$$X = \frac{5}{56} \begin{bmatrix} 13\,700 \\ 16\,536 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{17\,125}{14} \\ \frac{10\,335}{7} \end{bmatrix}$$

$$X = \begin{bmatrix} 1223.21 \\ 1476.43 \end{bmatrix}$$

e)

	X_{ij}		Y_i	X_i
	1	2		
1	244.64	590.57	388	1223.21
2	366.96	221.46	888	1476.43

$$X_{11} = a_{11} X_1 = \frac{1}{5} \cdot 1223.21 = 244.642 \approx 244.64$$

$$X_{12} = a_{12} X_2 = \frac{2}{5} \cdot 1476.43 = 590.572 \approx 590.57$$

$$X_{21} = a_{21} X_1 = \frac{3}{10} \cdot 1223.21 = 366.963 \approx 366.96$$

$$X_{22} = a_{22} X_2 = \frac{3}{20} \cdot 1476.43 = 221.4645 \approx 221.46$$

$$a_{ij} = \frac{X_{ij}}{X_j} \qquad X_{ij} = a_{ij} X_j$$

$$A = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{3}{10} & \frac{3}{20} \end{bmatrix}$$

$$X = \begin{bmatrix} 1223.21 \\ 1476.43 \end{bmatrix}$$

$$Y = \begin{bmatrix} 388 \\ 888 \end{bmatrix}$$

Zadatak 2

Zadana je matrica tehnologije

$$\begin{bmatrix} 1 & 0 & -0.5 \\ -0.1 & 1 & 0 \\ -0.2 & -0.5 & 1 \end{bmatrix}.$$

a) *Odredite matricu tehničkih koeficijenata i matricu finalne potražnje ako je zadana matrica ukupnog outputa pojedinog sektora*

$$\begin{bmatrix} 750 \\ 340 \\ 420 \end{bmatrix}.$$

b) *Napravite pripadnu input-output tablicu.*

c) *Napravite novu međusektorsku tablicu tako da finalna potražnja bude vektor* $(800, 350, 400)$.

Rješenje

a) Matrica tehnologije $T = I - A$

$$T = \begin{bmatrix} 1 & 0 & -0.5 \\ -0.1 & 1 & 0 \\ -0.2 & -0.5 & 1 \end{bmatrix}$$

Matrica tehničkih koeficijenata

$$A = I - T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -0.5 \\ -0.1 & 1 & 0 \\ -0.2 & -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.5 \\ 0.1 & 0 & 0 \\ 0.2 & 0.5 & 0 \end{bmatrix}$$

$(I - A)X = Y$

Matrica finalne potražnje $Y = (\overbrace{I - A}^{T})X$

$$Y = \begin{bmatrix} 1 & 0 & -0.5 \\ -0.1 & 1 & 0 \\ -0.2 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} 750 \\ 340 \\ 420 \end{bmatrix}$$

$$Y = \begin{bmatrix} 540 \\ 265 \\ 100 \end{bmatrix}$$

b)

	X_{ij}			Y_i	X_i
	1	2	3		
1	0	0	210	540	750
2	75	0	0	265	340
3	150	170	0	100	420

$$A = \begin{bmatrix} 0 & 0 & 0.5 \\ 0.1 & 0 & 0 \\ 0.2 & 0.5 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 750 \\ 340 \\ 420 \end{bmatrix} \qquad Y = \begin{bmatrix} 540 \\ 265 \\ 100 \end{bmatrix}$$

$X_{11} = a_{11}X_1 = 0 \cdot 750 = 0$

$X_{12} = a_{12}X_2 = 0 \cdot 340 = 0$

$X_{13} = a_{13}X_3 = 0.5 \cdot 420 = 210$

$X_{21} = a_{21}X_1 = 0.1 \cdot 750 = 75$

$X_{22} = a_{22}X_2 = 0 \cdot 340 = 0$

$X_{23} = a_{23}X_3 = 0 \cdot 420 = 0$

$X_{31} = a_{31}X_1 = 0.2 \cdot 750 = 150$

$X_{32} = a_{32}X_2 = 0.5 \cdot 340 = 170$

$X_{33} = a_{33}X_3 = 0 \cdot 420 = 0$

$a_{ij} = \dfrac{X_{ij}}{X_j}$ $\quad X_{ij} = a_{ij}X_j$

c) Matrica multiplikator $M = (I - A)^{-1} = T^{-1}$ $(I - A)X = Y$

$$T = I - A = \begin{bmatrix} 1 & 0 & -0.5 \\ -0.1 & 1 & 0 \\ -0.2 & -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ -\frac{1}{10} & 1 & 0 \\ -\frac{1}{5} & -\frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{vmatrix} 10 & 0 & -5 \\ -1 & 10 & 0 \\ -2 & -5 & 10 \end{vmatrix} = 875$$

$$I - A = \frac{1}{10} \begin{bmatrix} 10 & 0 & -5 \\ -1 & 10 & 0 \\ -2 & -5 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 10 & 0 & -5 \\ -1 & 10 & 0 \\ -2 & -5 & 10 \end{bmatrix}$$

$B^{-1} = \frac{1}{\det B} \cdot B^*$

$$(I - A)^{-1} = \left(\frac{1}{10}\right)^{-1} \begin{bmatrix} 10 & 0 & -5 \\ -1 & 10 & 0 \\ -2 & -5 & 10 \end{bmatrix}^{-1} = 10B^{-1}$$

$$B = \begin{bmatrix} 10 & 0 & -5 \\ -1 & 10 & 0 \\ -2 & -5 & 10 \end{bmatrix}$$

$A_{ij} = (-1)^{i+j} M_{ij}$

$$B_{11} = (-1)^{1+1} \begin{vmatrix} 10 & 0 \\ -5 & 10 \end{vmatrix} = 100 \qquad B_{22} = (-1)^{2+2} \begin{vmatrix} 10 & -5 \\ -2 & 10 \end{vmatrix} = 90$$

$$B_{12} = (-1)^{1+2} \begin{vmatrix} -1 & 0 \\ -2 & 10 \end{vmatrix} = 10 \qquad B_{23} = (-1)^{2+3} \begin{vmatrix} 10 & 0 \\ -2 & -5 \end{vmatrix} = 50$$

$$B_{13} = (-1)^{1+3} \begin{vmatrix} -1 & 10 \\ -2 & -5 \end{vmatrix} = 25 \qquad B_{31} = (-1)^{3+1} \begin{vmatrix} 0 & -5 \\ 10 & 0 \end{vmatrix} = 50$$

$$B_{21} = (-1)^{2+1} \begin{vmatrix} 0 & -5 \\ -5 & 10 \end{vmatrix} = 25 \qquad B_{32} = (-1)^{3+2} \begin{vmatrix} 10 & -5 \\ -1 & 0 \end{vmatrix} = 5$$

$$B_{33} = (-1)^{3+3} \begin{vmatrix} 10 & 0 \\ -1 & 10 \end{vmatrix} = 100$$

$$B = \begin{bmatrix} 10 & 0 & -5 \\ -1 & 10 & 0 \\ -2 & -5 & 10 \end{bmatrix}$$

$$B_{11} = 100, \quad B_{12} = 10, \quad B_{13} = 25$$
$$B_{21} = 25, \quad B_{22} = 90, \quad B_{23} = 50$$
$$B_{31} = 50, \quad B_{32} = 5, \quad B_{33} = 100$$

$\det B = 875$

$$B^{-1} = \frac{1}{875} \begin{bmatrix} 100 & 10 & 25 \\ 25 & 90 & 50 \\ 50 & 5 & 100 \end{bmatrix}^T = \frac{1}{875} \begin{bmatrix} 100 & 25 & 50 \\ 10 & 90 & 5 \\ 25 & 50 & 100 \end{bmatrix}$$

$$(I - A)^{-1} = 10B^{-1} = 10 \cdot \frac{1}{875} \begin{bmatrix} 100 & 25 & 50 \\ 10 & 90 & 5 \\ 25 & 50 & 100 \end{bmatrix} = \frac{2}{175} \begin{bmatrix} 100 & 25 & 50 \\ 10 & 90 & 5 \\ 25 & 50 & 100 \end{bmatrix}$$

$B^{-1} = \frac{1}{\det B} \cdot B^*$

$$(I - A)X = Y$$
$$X = (I - A)^{-1} Y$$

$$X = \frac{2}{175} \begin{bmatrix} 100 & 25 & 50 \\ 10 & 90 & 5 \\ 25 & 50 & 100 \end{bmatrix} \begin{bmatrix} 800 \\ 350 \\ 400 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{8\,700}{7} \\ \frac{3\,320}{7} \\ \frac{6\,200}{7} \end{bmatrix}$$

$$X = \frac{2}{175} \begin{bmatrix} 108\,750 \\ 41\,500 \\ 77\,500 \end{bmatrix}$$

$$X = \begin{bmatrix} 1242.86 \\ 474.29 \\ 885.71 \end{bmatrix}$$

	X_{ij}			Y_i	X_i
	1	2	3		
1	0	0	442.86	800	1242.86
2	124.29	0	0	350	474.29
3	248.57	237.14	0	400	885.71

$$A = \begin{bmatrix} 0 & 0 & 0.5 \\ 0.1 & 0 & 0 \\ 0.2 & 0.5 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 800 \\ 350 \\ 400 \end{bmatrix} \qquad X = \begin{bmatrix} 1242.86 \\ 474.29 \\ 885.71 \end{bmatrix}$$

$X_{11} = a_{11}X_1 = 0 \cdot 1242.86 = 0$

$X_{12} = a_{12}X_2 = 0 \cdot 474.29 = 0$

$X_{13} = a_{13}X_3 = 0.5 \cdot 885.71 = 442.86$

$X_{21} = a_{21}X_1 = 0.1 \cdot 1242.86 = 124.29$

$X_{22} = a_{22}X_2 = 0 \cdot 474.29 = 0$

$X_{23} = a_{23}X_3 = 0 \cdot 885.71 = 0$

$X_{31} = a_{31}X_1 = 0.2 \cdot 1242.86 = 248.57$

$X_{32} = a_{32}X_2 = 0.5 \cdot 474.29 = 237.14$

$X_{33} = a_{33}X_3 = 0 \cdot 885.71 = 0$

$$a_{ij} = \frac{X_{ij}}{X_j} \qquad X_{ij} = a_{ij}X_j$$

Rješenje

a) Matrica multiplikator

$$M = (I - A)^{-1}, \quad M = T^{-1}$$

$$M = \begin{bmatrix} 3.2 & 3.2 \\ 1.75 & 3.5 \end{bmatrix}$$

Matrica tehnologije

$$T = I - A, \quad T = M^{-1}$$

$\det M = 3.2 \cdot 3.5 - 1.75 \cdot 3.2 = 5.6$

$$T = \frac{1}{5.6}\begin{bmatrix} 3.5 & -3.2 \\ -1.75 & 3.2 \end{bmatrix} = \begin{bmatrix} \frac{5}{8} & -\frac{4}{7} \\ -\frac{5}{16} & \frac{4}{7} \end{bmatrix}$$

$$\frac{-1.75}{5.6} = \frac{-175}{560} = \frac{-5}{16}$$

Matrica tehničkih koeficijenata

$$A = I - T$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{5}{8} & -\frac{4}{7} \\ -\frac{5}{16} & \frac{4}{7} \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{3}{8} & \frac{4}{7} \\ \frac{5}{16} & \frac{3}{7} \end{bmatrix}$$

$$\frac{3.5}{5.6} = \frac{35}{56} = \frac{5}{8}$$

$$\frac{-3.2}{5.6} = \frac{-32}{56} = \frac{-4}{7}$$

Zadatak 3

Zadana je matrica multiplikator

$$\begin{bmatrix} 3.2 & 3.2 \\ 1.75 & 3.5 \end{bmatrix}.$$

a) *Napravite input-output tablicu ako je vektor finalne potražnje jednak* $(130, 95)$.

b) *Za koliko posto se mora povećati proizvodnja u pojedinom sektoru ako se finalna potražnja u prvom sektoru poveća za* 20%*?*

$$(I - A)X = Y$$

$$X = (I - A)^{-1}Y$$

$$X = \begin{bmatrix} 3.2 & 3.2 \\ 1.75 & 3.5 \end{bmatrix}\begin{bmatrix} 130 \\ 95 \end{bmatrix}$$

$$X = \begin{bmatrix} 720 \\ 560 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{3}{8} & \frac{4}{7} \\ \frac{5}{16} & \frac{3}{7} \end{bmatrix}$$

	X_{ij}		Y_i	X_i
	1	2		
1	270	320	130	720
2	225	240	95	560

$X_{11} = a_{11}X_1 = \frac{3}{8} \cdot 720 = 270$

$X_{12} = a_{12}X_2 = \frac{4}{7} \cdot 560 = 320$

$X_{21} = a_{21}X_1 = \frac{5}{16} \cdot 720 = 225$

$X_{22} = a_{22}X_2 = \frac{3}{7} \cdot 560 = 240$

$$a_{ij} = \frac{X_{ij}}{X_j} \qquad X_{ij} = a_{ij}X_j$$

b) $Y = \begin{bmatrix} 130 \\ 95 \end{bmatrix} \begin{matrix} \longleftarrow Y_1 \\ \longleftarrow Y_2 \end{matrix}$ $X = \begin{bmatrix} 720 \\ 560 \end{bmatrix} \begin{matrix} \longleftarrow X_1 \\ \longleftarrow X_2 \end{matrix}$ $Y' = \begin{bmatrix} 156 \\ 95 \end{bmatrix} \begin{matrix} \longleftarrow Y_1' \\ \longleftarrow Y_2' \end{matrix}$

Y_1 se poveća za 20%

$Y_1' = 130 + \frac{20}{100} \cdot 130$

$Y_1' = 130 + 26 = 156$

$(I - A)X' = Y'$

$X' = (I - A)^{-1} Y'$

$X' = \begin{bmatrix} 3.2 & 3.2 \\ 1.75 & 3.5 \end{bmatrix} \begin{bmatrix} 156 \\ 95 \end{bmatrix}$

$X' = \begin{bmatrix} 803.2 \\ 605.5 \end{bmatrix} \begin{matrix} \longleftarrow X_1' \\ \longleftarrow X_2' \end{matrix}$

Prvi sektor

$X_1 = 720,\ X_1' = 803.2,\ X_1' - X_1 = 83.2$

$$p_1 = \frac{100 \cdot 83.2}{720} \approx 11.56$$

Proizvodnja se mora povećati za 11.56%.

Drugi sektor

$X_2 = 560,\ X_2' = 605.5,\ X_2' - X_2 = 45.5$

$$p_2 = \frac{100 \cdot 45.5}{560} = 8.125$$

Proizvodnja se mora povećati za 8.125%.

$y = \frac{p}{100} \cdot x$ $p = \frac{100y}{x}$

$(I - A)X = Y$

Rješenje

a) Matrica tehničkih koeficijenata

$$A = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.2 \end{bmatrix}$$

Matrica tehnologije

$$I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.1 & 0 \\ 0 & 0.7 & -0.1 \\ -0.1 & -0.1 & 0.8 \end{bmatrix}$$

Zadatak 4

Zadana je matrica tehničkih koeficijenata

$$\begin{bmatrix} 0.2 & 0.1 & 0 \\ 0 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.2 \end{bmatrix}$$

jednog trosektorskog modela ekonomije.

a) *Koliko je od proizvodnje prvog i trećeg sektora namijenjeno zadovoljenju finalne potražnje ako je ukupna proizvodnja u prvom sektoru jednaka* 140 *jedinica, u trećem* 200 *jedinica, a od proizvodnje drugog sektora za finalnu potražnju se odvaja* 85 *jedinica?*

b) *Koliko se od proizvodnje pojedinog sektora troši u procesu reprodukcije?*

$X_1 = 140$

$X_3 = 200$

$Y_2 = 85$

$(I - A)X = Y$

$$\begin{bmatrix} 0.8 & -0.1 & 0 \\ 0 & 0.7 & -0.1 \\ -0.1 & -0.1 & 0.8 \end{bmatrix} \begin{bmatrix} 140 \\ X_2 \\ 200 \end{bmatrix} = \begin{bmatrix} Y_1 \\ 85 \\ Y_3 \end{bmatrix}$$

$$\begin{matrix} 0.8 \cdot 140 - 0.1 \cdot X_2 + 0 \cdot 200 \\ 0 \cdot 140 + 0.7 \cdot X_2 - 0.1 \cdot 200 \\ -0.1 \cdot 140 - 0.1 \cdot X_2 + 0.8 \cdot 200 \end{matrix} \begin{bmatrix} 112 - 0.1X_2 \\ 0.7X_2 - 20 \\ 146 - 0.1X_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ 85 \\ Y_3 \end{bmatrix}$$

$112 - 0.1X_2 = Y_1 \rightsquigarrow Y_1 = 112 - 0.1 \cdot 150$ $\quad Y_1 = 97$

$0.7X_2 - 20 = 85 \longrightarrow 0.7X_2 = 105$ $\quad X_2 = 150$

$146 - 0.1X_2 = Y_3 \rightsquigarrow Y_3 = 146 - 0.1 \cdot 150$ $\quad Y_3 = 131$

b)

$X_1 = \boxed{X_{11} + X_{12} + X_{13}} + Y_1$

dio proizvodnje prvog sektora koji se troši u procesu reprodukcije

$X_1 - Y_1 = 140 - 97 = 43$

Od ukupne proizvodnje prvog sektora u procesu reprodukcije troše se 43 jedinice.

$X_2 = \boxed{X_{21} + X_{22} + X_{23}} + Y_2$

dio proizvodnje drugog sektora koji se troši u procesu reprodukcije

$X_2 - Y_2 = 150 - 85 = 65$

Od ukupne proizvodnje drugog sektora u procesu reprodukcije troši se 65 jedinica.

$X_3 = \boxed{X_{31} + X_{32} + X_{33}} + Y_3$

dio proizvodnje trećeg sektora koji se troši u procesu reprodukcije

$X_3 - Y_3 = 200 - 131 = 69$

Od ukupne proizvodnje trećeg sektora u procesu reprodukcije troši se 69 jedinica.

Domaća zadaća

Napravite međusektorsku tablicu.

	X_{ij}			Y_i	X_i
	1	2	3		
1	28	15	0	97	140
2	0	45	20	85	150
3	14	15	40	131	200