Inverzna matrica i matrične jednadžbe

MATEMATIKA ZA EKONOMISTE 1

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Inverzna matrica

- Inverzna matrica kvadratne matrice A je matrica A^{-1} za koju vrijedi

$$AA^{-1} = A^{-1}A = I$$

- Formula za određivanje inverzne matrice

$$A^{-1} = \frac{1}{\det A} \cdot A^*$$

adjunkta matrice A

- kvadratna matrica reda 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$A_{ij} = (-1)^{i+1} M_{ij}$

$$A^* = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

- kvadratna matrica reda 2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Zadatak 1

Odredite inverzne matrice sljedećih matrica:

a) $A = \begin{bmatrix} 2 & 1 \\ -5 & 4 \end{bmatrix}$

b) $B = \frac{2}{3} \begin{bmatrix} 3 & 5 \\ 1 & -3 \end{bmatrix}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Rješenje

a) $\det A = 2 \cdot 4 - (-5) \cdot 1 = 8 + 5 = 13$

$$A^{-1} = \frac{1}{13} \begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix}$$

b) $(kA)^{-1} = k^{-1}A^{-1}$

$$B^{-1} = \left(\frac{2}{3}\right)^{-1} \cdot \begin{bmatrix} 3 & 5 \\ 1 & -3 \end{bmatrix}^{-1} =$$

$$= \frac{3}{2} \cdot \frac{1}{-14} \begin{bmatrix} -3 & -5 \\ -1 & 3 \end{bmatrix} =$$

$$= -\frac{3}{28} \begin{bmatrix} -3 & -5 \\ -1 & 3 \end{bmatrix}$$

Zadatak 2

Odredite inverznu matricu matrice

$$A = \frac{5}{3} \begin{bmatrix} 1 & 5 & 8 \\ 2 & 3 & -1 \\ 4 & -5 & -9 \end{bmatrix}.$$

$$B = \begin{bmatrix} 1 & 5 & 8 \\ 2 & 3 & -1 \\ 4 & -5 & -9 \end{bmatrix}$$

$$\det B = \begin{vmatrix} 1 & 5 & 8 \\ 2 & 3 & -1 \\ 4 & -5 & -9 \end{vmatrix} = -138$$

Rješenje

$$A^{-1} = \left(\frac{5}{3}\right)^{-1} \begin{bmatrix} 1 & 5 & 8 \\ 2 & 3 & -1 \\ 4 & -5 & -9 \end{bmatrix}^{-1} = \frac{3}{5} B^{-1}$$

$(kA)^{-1} = k^{-1}A^{-1}$

$$B^{-1} = \frac{1}{\det B} \cdot B^*$$

$$B = \begin{bmatrix} 1 & 5 & 8 \\ 2 & 3 & -1 \\ 4 & -5 & -9 \end{bmatrix}$$

$A_{ij} = (-1)^{i+j} M_{ij}$

$$B_{11} = (-1)^{1+1} \begin{vmatrix} 3 & -1 \\ -5 & -9 \end{vmatrix} = -32 \qquad B_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 8 \\ 4 & -9 \end{vmatrix} = -41$$

$$B_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -1 \\ 4 & -9 \end{vmatrix} = 14 \qquad B_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 5 \\ 4 & -5 \end{vmatrix} = 25$$

$$B_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = -22 \qquad B_{31} = (-1)^{3+1} \begin{vmatrix} 5 & 8 \\ 3 & -1 \end{vmatrix} = -29$$

$$B_{21} = (-1)^{2+1} \begin{vmatrix} 5 & 8 \\ -5 & -9 \end{vmatrix} = 5 \qquad B_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 8 \\ 2 & -1 \end{vmatrix} = 17$$

$$B_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} = -7$$

$$B = \begin{bmatrix} 1 & 5 & 8 \\ 2 & 3 & -1 \\ 4 & -5 & -9 \end{bmatrix}$$

$$B_{11} = -32, \quad B_{12} = 14, \quad B_{13} = -22$$
$$B_{21} = 5, \quad B_{22} = -41, \quad B_{23} = 25$$
$$B_{31} = -29, \quad B_{32} = 17, \quad B_{33} = -7$$

$\det B = -138$

$$B^{-1} = \frac{1}{-138} \begin{bmatrix} -32 & 14 & -22 \\ 5 & -41 & 25 \\ -29 & 17 & -7 \end{bmatrix}^T = -\frac{1}{138} \begin{bmatrix} -32 & 5 & -29 \\ 14 & -41 & 17 \\ -22 & 25 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{3}{5} B^{-1} = \frac{3}{5} \cdot \frac{-1}{138} \begin{bmatrix} -32 & 5 & -29 \\ 14 & -41 & 17 \\ -22 & 25 & -7 \end{bmatrix} = \frac{-1}{230} \begin{bmatrix} -32 & 5 & -29 \\ 14 & -41 & 17 \\ -22 & 25 & -7 \end{bmatrix}$$

$$B^{-1} = \frac{1}{\det B} \cdot B^*$$

Zadatak 3

Neka je A regularna matrica reda 4 i $\det A = \frac{1}{4}$. Izračunajte:

a) $\det\left(\frac{1}{2}\left(A^{-1}\right)^3\right)$ b) $\det\left(A^{-1}A^T\right)^{-2}$

Rješenje $\det(A^m) = (\det A)^m$ $\det(AB) = \det A \det B$

a)
$$\det\left(\frac{1}{2}\left(A^{-1}\right)^3\right) = \left(\frac{1}{2}\right)^4 \cdot \det\left(A^{-1}\right)^3 = \frac{1}{16} \cdot \left(\det A^{-1}\right)^3 =$$
$$= \frac{1}{16} \cdot \left(\frac{1}{\det A}\right)^3 = \frac{1}{16} \cdot 4^3 = 4$$

b)
$$\det\left(A^{-1}A^T\right)^{-2} = \left(\det\left(A^{-1}A^T\right)\right)^{-2} = \left(\det A^{-1} \det A^T\right)^{-2} =$$
$$= \left(\frac{1}{\det A} \cdot \det A\right)^{-2} = 1^{-2} = 1$$

$\det(kA) = k^n \det A$ n je red kvadratne matrice A

Napomena

$$A^{-n} = \left(A^{-1}\right)^n, \quad n \in \mathbb{N}$$

Na primjer,

$$A^{-3} = \left(A^{-1}\right)^3$$

Matrične jednadžbe

- Neka je A kvadratna matrica i $\det A \neq 0$.

$$
\begin{aligned}
A^{-1} \cdot / \; AX &= B \\
A^{-1}(AX) &= A^{-1}B \\
(A^{-1}A)X &= A^{-1}B \\
I \cdot X &= A^{-1}B \\
X &= A^{-1}B
\end{aligned}
\qquad
\begin{aligned}
XA &= B \; / \cdot A^{-1} \\
(XA)A^{-1} &= BA^{-1} \\
X(AA^{-1}) &= BA^{-1} \\
X \cdot I &= BA^{-1} \\
X &= BA^{-1}
\end{aligned}
$$

Zadatak 4

Riješite sljedeće matrične jednadžbe:

a) $AX = BX + C$

b) $AXB - 2XB = B + AB$

c) $B(XA - I) + B^2 + 2BXA = O$

d) $A^{-1}(I + A^2X)B = 2AX$

Rješenje

a)

$$
\begin{aligned}
AX &= BX + C \\
AX - BX &= C \\
(A - B)^{-1} \cdot / \; (A - B)X &= C \\
X &= (A - B)^{-1}C
\end{aligned}
$$

b)

$$
\begin{aligned}
AXB - 2XB &= B + AB \\
(A - 2I)^{-1} \cdot / \; (A - 2I)XB &= (I + A)B \\
XB &= (A - 2I)^{-1}(I + A)B \; / \cdot B^{-1} \\
X &= (A - 2I)^{-1}(I + A)
\end{aligned}
$$

c)

$$
\begin{aligned}
B(XA - I) + B^2 + 2BXA &= O \\
BXA - B + B^2 + 2BXA &= O \\
3BXA &= B - B^2 \; / \cdot \tfrac{1}{3} \\
B^{-1} \cdot / \; BXA &= \tfrac{1}{3}(B - B^2) \\
XA &= B^{-1} \cdot \tfrac{1}{3}(B - B^2) \\
XA &= \tfrac{1}{3}(I - B) \; / \cdot A^{-1} \\
X &= \tfrac{1}{3}(I - B)A^{-1}
\end{aligned}
$$

d)

$$A^{-1}(I + A^2X)B = 2AX$$

$$\left(A^{-1} + AX\right)B = 2AX$$

$$A^{-1}B + AXB = 2AX$$

$$AXB - 2AX = -A^{-1}B$$

$$AXB - AX \cdot 2I = -A^{-1}B$$

$$A^{-1} \cdot / \; AX(B - 2I) = -A^{-1}B$$

$$X(B - 2I) = A^{-1} \cdot (-A^{-1}B)$$

$$X(B - 2I) = -A^{-2}B \; / \cdot (B - 2I)^{-1}$$

$$X = -A^{-2}B(B - 2I)^{-1}$$

Zadatak 5

Riješite matričnu jednadžbu $X - I = ABX$ ako je

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} -3 & -2 \\ -3 & 1 \end{bmatrix}.$$

Rješenje

$$X - I = ABX$$

$$X - ABX = I$$

$$(I - AB)^{-1} \cdot / \; (I - AB)X = I$$

$$X = (I - AB)^{-1} \cdot I$$

$$\boxed{X = (I - AB)^{-1}}$$

$\boxed{X = (I - AB)^{-1}}$

$$AB = \begin{bmatrix} 1 & -2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 9 & 6 \end{bmatrix}$$

$$I - AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -4 \\ 9 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -9 & -5 \end{bmatrix}$$

$$\det(I - AB) = -2 \cdot (-5) - (-9) \cdot 4 = 10 + 36 = 46$$

$$X = (I - AB)^{-1} = \frac{1}{46} \begin{bmatrix} -5 & -4 \\ 9 & -2 \end{bmatrix}$$