Determinante

MATEMATIKA ZA EKONOMISTE 1

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Determinante

Zadatak 1

Izračunajte sljedeće determinante:

a) $\begin{vmatrix} 2 & 5 \\ 1 & -3 \end{vmatrix}$, b) $\begin{vmatrix} 2 & -5 \\ 1 & -3 \end{vmatrix}$, c) $\begin{vmatrix} x-a & -a \\ a & x+a \end{vmatrix}$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Rješenje

$$\begin{vmatrix} 2 & 5 \\ 1 & -3 \end{vmatrix} = 2 \cdot (-3) - 1 \cdot 5 = -6 - 5 = -11$$

$$\begin{vmatrix} 2 & -5 \\ 1 & -3 \end{vmatrix} = 2 \cdot (-3) - 1 \cdot (-5) = -6 + 5 = -1$$

$$\begin{vmatrix} x-a & -a \\ a & x+a \end{vmatrix} = (x-a)(x+a) - a \cdot (-a) = x^2 - a^2 + a^2 = x^2$$

Zadatak 2

Izračunajte determinantu

$$\begin{vmatrix} 9 & 4 & -5 \\ 8 & 7 & -2 \\ 2 & -1 & 8 \end{vmatrix}$$

a) *Sarrusovim pravilom,*

b) *svođenjem na trokutastu determinantu,*

c) *Laplaceovim razvojem po trećem stupcu,*

d) *Laplaceovim razvojem po prvom retku.*

Rješenje

a)

$$\begin{vmatrix} 9 & 4 & -5 \\ 8 & 7 & -2 \\ 2 & -1 & 8 \end{vmatrix}\begin{matrix} 9 & 4 \\ 8 & 7 \\ 2 & -1 \end{matrix} =$$

$$= 9 \cdot 7 \cdot 8 + 4 \cdot (-2) \cdot 2 + (-5) \cdot 8 \cdot (-1) - 2 \cdot 7 \cdot (-5) -$$

$$- (-1) \cdot (-2) \cdot 9 - 8 \cdot 8 \cdot 4 =$$

$$= 504 - 16 + 40 + 70 - 18 - 256 = 324$$

b)

$$\begin{vmatrix} 9 & 4 & -5 \\ 8 & 7 & -2 \\ 2 & -1 & 8 \end{vmatrix} = -\begin{vmatrix} 2 & -1 & 8 \\ 8 & 7 & -2 \\ 9 & 4 & -5 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 8 \\ 7 & 8 & -2 \\ 4 & 9 & -5 \end{vmatrix} \begin{matrix} / \cdot 7 \ / \cdot 4 \\ + \\ + \end{matrix} =$$

$$= \begin{vmatrix} -1 & 2 & 8 \\ 0 & 22 & 54 \\ 0 & 17 & 27 \end{vmatrix} \begin{matrix} \\ / \cdot \frac{-17}{22} \\ + \end{matrix} = \begin{vmatrix} -1 & 2 & 8 \\ 0 & 22 & 54 \\ 0 & 0 & \frac{-162}{11} \end{vmatrix} = -1 \cdot 22 \cdot \frac{-162}{11} = 324$$

$$\boxed{A_{ij} = (-1)^{i+j} M_{ij}}$$

c)

$$\begin{vmatrix} 9 & 4 & -5 \\ 8 & 7 & -2 \\ 2 & -1 & 8 \end{vmatrix} = -5 \cdot A_{13} + (-2) \cdot A_{23} + 8 \cdot A_{33} =$$

$$= -5 \cdot (-1)^{1+3} \begin{vmatrix} 8 & 7 \\ 2 & -1 \end{vmatrix} + (-2) \cdot (-1)^{2+3} \begin{vmatrix} 9 & 4 \\ 2 & -1 \end{vmatrix} + 8 \cdot (-1)^{3+3} \begin{vmatrix} 9 & 4 \\ 8 & 7 \end{vmatrix} =$$

$$= -5 \cdot 1 \cdot (-22) + (-2) \cdot (-1) \cdot (-17) + 8 \cdot 1 \cdot 31 =$$

$$= 110 - 34 + 248 = 324$$

$$\boxed{A_{ij} = (-1)^{i+j} M_{ij}}$$

d)

$$\begin{vmatrix} 9 & 4 & -5 \\ 8 & 7 & -2 \\ 2 & -1 & 8 \end{vmatrix} = 9 \cdot A_{11} + 4 \cdot A_{12} + (-5) \cdot A_{13} =$$

$$= 9 \cdot (-1)^{1+1} \begin{vmatrix} 7 & -2 \\ -1 & 8 \end{vmatrix} + 4 \cdot (-1)^{1+2} \begin{vmatrix} 8 & -2 \\ 2 & 8 \end{vmatrix} + (-5) \cdot (-1)^{1+3} \begin{vmatrix} 8 & 7 \\ 2 & -1 \end{vmatrix}$$

$$= 9 \cdot 1 \cdot 54 + 4 \cdot (-1) \cdot 68 + (-5) \cdot 1 \cdot (-22) =$$

$$= 486 - 272 + 110 = 324$$

Zadatak 3

Izračunajte determinantu

$$\begin{vmatrix}2&-5&1&2\\-3&7&-1&4\\5&-9&2&7\\4&-6&1&2\end{vmatrix}.$$

Rješenje

1. način: svođenje na trokutastu matricu

$$\begin{vmatrix}2&-5&1&2\\-3&7&-1&4\\5&-9&2&7\\4&-6&1&2\end{vmatrix} = -\begin{vmatrix}1&-5&2&2\\-1&7&-3&4\\2&-9&5&7\\1&-6&4&2\end{vmatrix} \begin{matrix}/\cdot 1 & /\cdot(-2) & /\cdot(-1)\\ & & \\ & & \\ & & \end{matrix} =$$

$$= -\begin{vmatrix}1&-5&2&2\\0&2&-1&6\\0&1&1&3\\0&-1&2&0\end{vmatrix} = \begin{vmatrix}1&2&-5&2\\0&-1&2&6\\0&1&1&3\\0&2&-1&0\end{vmatrix} \begin{matrix} & \\ /\cdot 1 & /\cdot 2\\ & \\ & \end{matrix} =$$

$$= \begin{vmatrix}1&2&-5&2\\0&-1&2&6\\0&0&3&9\\0&0&3&12\end{vmatrix} \begin{matrix} \\ \\ /\cdot(-1) \\ \\ \end{matrix} = \begin{vmatrix}1&2&-5&2\\0&-1&2&6\\0&0&3&9\\0&0&0&3\end{vmatrix} = 1\cdot(-1)\cdot 3\cdot 3 = -9$$

$A_{ij} = (-1)^{i+j} M_{ij}$

2. način: Laplaceov razvoj, npr. po 3. retku

$$\begin{vmatrix}2&-5&1&2\\-3&7&-1&4\\5&-9&2&7\\4&-6&1&2\end{vmatrix} = 5\cdot A_{31} + (-9)\cdot A_{32} + 2\cdot A_{33} + 7\cdot A_{34} =$$

$$= 5\cdot(-1)^{3+1}\underbrace{\begin{vmatrix}-5&1&2\\7&-1&4\\-6&1&2\end{vmatrix}}_{=-6} + (-9)\cdot(-1)^{3+2}\underbrace{\begin{vmatrix}2&1&2\\-3&-1&4\\4&1&2\end{vmatrix}}_{=12} +$$

$$+ 2\cdot(-1)^{3+3}\underbrace{\begin{vmatrix}2&-5&2\\-3&7&4\\4&-6&2\end{vmatrix}}_{=-54} + 7\cdot(-1)^{3+4}\underbrace{\begin{vmatrix}2&-5&1\\-3&7&-1\\4&-6&1\end{vmatrix}}_{=-3} = \overset{\text{DZ}}{\cdots} = -9$$

$A_{ij} = (-1)^{i+j} M_{ij}$

3. način: svojstva determinanti i Laplaceov razvoj

$$\begin{vmatrix}2&-5&1&2\\-3&7&-1&4\\5&-9&2&7\\4&-6&1&2\end{vmatrix} \begin{matrix}/\cdot 1 & /\cdot(-2) & /\cdot(-1)\\ & & \\ & & \\ & & \end{matrix} = \begin{vmatrix}2&-5&1&2\\-1&2&0&6\\1&1&0&3\\2&-1&0&0\end{vmatrix} =$$

$$= 1\cdot A_{13} + 0\cdot A_{23} + 0\cdot A_{33} + 0\cdot A_{43} = 1\cdot A_{13} =$$

$$= 1\cdot(-1)^{1+3}\begin{vmatrix}-1&2&6\\1&1&3\\2&-1&0\end{vmatrix} = 1\cdot 1\cdot(-9) = -9$$

$A_{ij} = (-1)^{i+j} M_{ij}$

3. način: svojstva determinanti i Laplaceov razvoj

$$\begin{vmatrix} 2 & -5 & 1 & 2 \\ -3 & 7 & -1 & 4 \\ 5 & -9 & 2 & 7 \\ 4 & -6 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ -1 & 2 & -1 & 6 \\ 1 & 1 & 2 & 3 \\ 2 & -1 & 1 & 0 \end{vmatrix} =$$

$+$ $/\cdot 5$

$+$ $/\cdot(-2)$ $+$

$/\cdot(-2)$

$$= 0 \cdot A_{11} + 0 \cdot A_{12} + 1 \cdot A_{13} + 0 \cdot A_{14} = 1 \cdot A_{13} =$$

$$= 1 \cdot (-1)^{1+3} \begin{vmatrix} -1 & 2 & 6 \\ 1 & 1 & 3 \\ 2 & -1 & 0 \end{vmatrix} = 1 \cdot 1 \cdot (-9) = -9$$

Zadatak 4

Zadana je matrica

$$A = \begin{bmatrix} 4+x & 2 & 2 \\ 7 & x-1 & 2 \\ x+1 & 5 & 5 \end{bmatrix}.$$

a) *Odredite sve* $x \in \mathbb{R}$ *za koje je* $\det A = 0$.

b) *Za* $x = -1$ *izračunajte*

$$\det\left(A^T\right) + 5\det\left(A^3\right) - 2\det\left(\tfrac{1}{2}A\right).$$

Rješenje

a)

$$\left|\begin{array}{ccc|cc} 4+x & 2 & 2 & 4+x & 2 \\ 7 & x-1 & 2 & 7 & x-1 \\ x+1 & 5 & 5 & x+1 & 5 \end{array}\right. =$$

$$= (4+x)\cdot(x-1)\cdot 5 + 2\cdot 2\cdot(x+1) + 2\cdot 7\cdot 5 -$$

$$-(x+1)\cdot(x-1)\cdot 2 - 5\cdot 2\cdot(4+x) - 5\cdot 7\cdot 2 =$$

$$= \underline{\underline{20x}} - 20 + \underline{5x^2} - \underline{\underline{5x}} + \underline{\underline{4x}} + 4 + 70 - \underline{2x^2} + 2 - 40 - \underline{\underline{10x}} - 70 =$$

$$= 3x^2 + 9x - 54$$

$$ax^2 + bx + c = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$3x^2 + 9x - 54 = 0 \;/ : 3$$

$$x^2 + 3x - 18 = 0$$

$$x_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4\cdot 1\cdot(-18)}}{2\cdot 1}$$

$$x_{1,2} = \frac{-3 \pm \sqrt{9+72}}{2}$$

$$x_{1,2} = \frac{-3 \pm 9}{2}$$

$$\boxed{x_1 = 3, \quad x_2 = -6}$$

$\boxed{\det(A^m) = (\det A)^m}$

b) $x = -1, \quad \det A = 3x^2 + 9x - 54$

$$\det A = 3 \cdot (-1)^2 + 9 \cdot (-1) - 54 = -60$$

$$\det\left(A^T\right) + 5\det\left(A^3\right) - 2\det\left(\tfrac{1}{2}A\right) =$$

$$= \det A + 5 \cdot (\det A)^3 - 2 \cdot \left(\tfrac{1}{2}\right)^3 \det A =$$

$$= -60 + 5 \cdot (-60)^3 - 2 \cdot \tfrac{1}{8} \cdot (-60) = -1\,080\,045$$

$\boxed{\det(kA) = k^n \det A}$ $\quad$ n je red kvadratne matrice A

Zadatak 5

Neka su A i B kvadratne matrice reda 4 pri čemu je $\det A = \frac{1}{2}$ i $\det B = -1$. Odredite:

a) $\det\left(A^5 A^T\right)$

b) $\det\left(B^T \cdot 2A\right)^T$

c) $\det\left(2AB\right)^3$

Rješenje $\quad \boxed{\det(AB) = \det A \det B}$ $\quad \boxed{\det(A^m) = (\det A)^m}$

a)
$$\det\left(A^5 A^T\right) = \det\left(A^5\right) \cdot \det\left(A^T\right) = \left(\det A\right)^5 \cdot \det A =$$
$$= \left(\det A\right)^6 = \left(\tfrac{1}{2}\right)^6 = \frac{1}{64}$$

b)
$$\det\left(B^T \cdot 2A\right)^T = \det\left(B^T \cdot 2A\right) = \det\left(B^T\right) \cdot \det(2A) =$$
$$= \det B \cdot 2^4 \det A = 16 \det A \det B = 16 \cdot \tfrac{1}{2} \cdot (-1) = -8$$

c)
$$\det\left(2AB\right)^3 = \left(\det(2AB)\right)^3 = \left(2^4 \det(AB)\right)^3 =$$
$$= \left(16 \det A \det B\right)^3 = \left(16 \cdot \tfrac{1}{2} \cdot (-1)\right)^3 = (-8)^3 = -512$$

$\boxed{\det(kA) = k^n \det A}$ $\quad$ n je red kvadratne matrice A