Derivacija realne funkcije realne varijable

MATEMATIKA ZA EKONOMISTE 1

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Zadatak 1

Odredite derivacije funkcija $f(x) = \sqrt[5]{x^2}$ *i* $g(x) = \left(\frac{2}{5}\right)^x$.

Rješenje

$$f(x) = \sqrt[5]{x^2} \qquad f(x) = x^{\frac{2}{5}}$$

$$\boxed{\sqrt[n]{x^m} = x^{\frac{m}{n}}}$$

$$f'(x) = \frac{2}{5} x^{\frac{2}{5}-1} = \frac{2}{5} x^{-\frac{3}{5}} = \frac{2}{5\sqrt[5]{x^3}}$$

$$\boxed{(x^n)' = nx^{n-1}}$$

$$g(x) = \left(\frac{2}{5}\right)^x \qquad g'(x) = \left(\frac{2}{5}\right)^x \ln\frac{2}{5}$$

$$\boxed{(a^x)' = a^x \ln a}$$

$$\boxed{x^{-\frac{3}{5}} = \frac{1}{x^{\frac{3}{5}}} = \frac{1}{\sqrt[5]{x^3}}}$$

Zadatak 2

Odredite derivaciju funkcije $f(x) = \frac{3}{5}x^3 - \frac{7}{5}x^2 + \frac{9}{5}x + \frac{4}{5}$.

Rješenje

$$\begin{aligned}
f'(x) &= \left(\frac{3}{5}x^3 - \frac{7}{5}x^2 + \frac{9}{5}x + \frac{4}{5}\right)' = \\
&= \left(\frac{3}{5}x^3\right)' - \left(\frac{7}{5}x^2\right)' + \left(\frac{9}{5}x\right)' + \left(\frac{4}{5}\right)' = \\
&= \frac{3}{5}\cdot(x^3)' - \frac{7}{5}\cdot(x^2)' + \frac{9}{5}\cdot(x)' + 0 = \\
&= \frac{3}{5}\cdot 3x^2 - \frac{7}{5}\cdot 2x + \frac{9}{5}\cdot 1 = \frac{9}{5}x^2 - \frac{14}{5}x + \frac{9}{5}
\end{aligned}$$

$$\boxed{(cu)'(x) = c\cdot u'(x)} \quad \boxed{(c)' = 0} \quad \boxed{(x^n)' = nx^{n-1}}$$

$$\boxed{(u+v)'(x) = u'(x) + v'(x)} \quad \boxed{(u-v)'(x) = u'(x) - v'(x)}$$

Zadatak 3

$$\boxed{\frac{x^n}{x^m} = x^{n-m}} \quad \boxed{\sqrt[n]{x^m} = x^{\frac{m}{n}}}$$

Odredite derivaciju funkcije $y = \dfrac{4x^2}{3\sqrt[7]{x^4}}$.

$$\boxed{(cu)'(x) = c\cdot u'(x)}$$

Rješenje

$$\boxed{(x^n)' = nx^{n-1}}$$

$$y = \frac{4x^2}{3\sqrt[7]{x^4}} = \frac{4x^2}{3x^{\frac{4}{7}}} = \frac{4}{3}x^{2-\frac{4}{7}}$$

$$y = \frac{4}{3}x^{\frac{10}{7}}$$

$$y' = \frac{4}{3}\cdot\left(x^{\frac{10}{7}}\right)' = \frac{4}{3}\cdot\frac{10}{7}x^{\frac{10}{7}-1}$$

$$y' = \frac{40}{21}x^{\frac{3}{7}} = \frac{40}{21}\sqrt[7]{x^3}$$

$\boxed{(uv)'(x) = u'(x)\cdot v(x) + u(x)\cdot v'(x)}$ $\boxed{(x^n)' = nx^{n-1}}$ $\boxed{(e^x)' = e^x}$

Zadatak 4

Odredite derivaciju funkcije $y = xe^x$ u točki 0.

Rješenje

$$y' = (x)'\cdot e^x + x\cdot (e^x)'$$
$$y' = 1\cdot e^x + x\cdot e^x$$
$$y' = (1+x)e^x$$
$$y'(0) = (1+0)\cdot e^0$$
$$y'(0) = 1\cdot 1$$
$$y'(0) = 1$$

Zadatak 5

$\boxed{(u-v)'(x) = u'(x) - v'(x)}$ $\boxed{(u+v)'(x) = u'(x) + v'(x)}$

Odredite derivaciju funkcije $y = \dfrac{x^3-5}{x^3+5}$.

Rješenje

$$y' = \frac{(x^3-5)'\cdot(x^3+5) - (x^3-5)\cdot(x^3+5)'}{(x^3+5)^2} =$$
$$= \frac{3x^2\cdot(x^3+5) - (x^3-5)\cdot 3x^2}{(x^3+5)^2} =$$
$$= \frac{3x^5+15x^2-3x^5+15x^2}{(x^3+5)^2} = \frac{30x^2}{(x^3+5)^2}$$

$\boxed{\left(\frac{u}{v}\right)'(x) = \frac{u'(x)\cdot v(x) - u(x)\cdot v'(x)}{v(x)^2}}$ $\boxed{(x^n)' = nx^{n-1}}$ $\boxed{(c)' = 0}$

Zadatak 6

Odredite derivaciju funkcije $y = \sqrt[3]{x}\log_2 x$.

$\boxed{\sqrt[n]{x^m} = x^{\frac{m}{n}}}$ $\boxed{\frac{x^m}{x^n} = x^{m-n}}$

Rješenje

$$y' = \left(\sqrt[3]{x}\log_2 x\right)' = \left(x^{\frac{1}{3}}\log_2 x\right)' =$$
$$= \left(x^{\frac{1}{3}}\right)'\cdot\log_2 x + x^{\frac{1}{3}}\cdot(\log_2 x)' =$$
$$= \frac{1}{3}x^{-\frac{2}{3}}\log_2 x + x^{\frac{1}{3}}\cdot\frac{1}{x\ln 2} =$$
$$= \frac{1}{3}x^{-\frac{2}{3}}\log_2 x + \frac{x^{-\frac{2}{3}}}{\ln 2} =$$
$$= \left(\frac{1}{3}\log_2 x + \frac{1}{\ln 2}\right)x^{-\frac{2}{3}}$$

$\boxed{(x^n)' = nx^{n-1}}$ $\boxed{(\log_a x)' = \frac{1}{x\ln a}}$ $\boxed{(uv)'(x) = u'(x)\cdot v(x) + u(x)\cdot v'(x)}$

Zadatak 7

$\boxed{(u+v)'(x) = u'(x) + v'(x)}$

Odredite derivaciju funkcije $y = \frac{\ln x+1}{\ln x-1}$.

Rješenje

$$y' = \frac{(\ln x+1)'\cdot(\ln x-1) - (\ln x+1)\cdot(\ln x-1)'}{(\ln x-1)^2} =$$
$$= \frac{\frac{1}{x}\cdot(\ln x-1) - (\ln x+1)\cdot\frac{1}{x}}{(\ln x-1)^2} =$$
$$= \frac{\frac{1}{x}\ln x - \frac{1}{x} - \frac{1}{x}\ln x - \frac{1}{x}}{(\ln x-1)^2} = \frac{-\frac{2}{x}}{(\ln x-1)^2} =$$
$$= \frac{-2}{x(\ln x-1)^2}$$

$\boxed{(c)' = 0}$ $\boxed{(\ln x)' = \frac{1}{x}}$ $\boxed{\left(\frac{u}{v}\right)'(x) = \frac{u'(x)\cdot v(x) - u(x)\cdot v'(x)}{v(x)^2}}$

Zadatak 8

Odredite derivaciju funkcije $y = 10^x \log x + \ln 10$.

Rješenje

$$y' = \left(10^x \log x + \ln 10\right)' = \left(10^x \log x\right)' + \left(\ln 10\right)' =$$

$$= \left(10^x\right)' \cdot \log x + 10^x \cdot \left(\log x\right)' + 0 =$$

$$= 10^x \ln 10 \cdot \log x + 10^x \cdot \frac{1}{x \ln 10} =$$

$$= \left(\ln 10 \log x + \frac{1}{x \ln 10}\right) 10^x$$

$\boxed{(c)' = 0}$

$\boxed{(a^x)' = a^x \ln a}$

$\boxed{(\log_a x)' = \frac{1}{x \ln a}}$

$\boxed{(u+v)'(x) = u'(x) + v'(x)}$

$\boxed{(uv)'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)}$

$\boxed{\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}}$

Zadatak 9

Odredite derivaciju funkcije $y = e^{\sqrt{x}}$.

Rješenje

$$y' = e^{\sqrt{x}} \cdot \left(\sqrt{x}\right)' = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

$$y' = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$\boxed{\left(e^{\text{nešto}}\right)' = e^{\text{nešto}} \cdot (\text{nešto})'}$

$\boxed{(e^x)' = e^x}$

Zadatak 10

Odredite derivaciju funkcije $y = (x^2 + 3x - 5)^{20}$.

Rješenje

$$y' = 20\left(x^2 + 3x - 5\right)^{19} \cdot \left(x^2 + 3x - 5\right)' =$$

$$= 20\left(x^2 + 3x - 5\right)^{19} (2x + 3)$$

$\boxed{\left((\text{nešto})^n\right)' = n(\text{nešto})^{n-1} \cdot (\text{nešto})'}$

$\boxed{(x^n)' = nx^{n-1}}$

$\boxed{(a^x)' = a^x \ln a}$ $\boxed{(x^n)' = nx^{n-1}}$

Zadatak 11

Odredite derivaciju funkcije $y = \log_5 (5^x - x^5)$.

Rješenje

$$y' = \frac{1}{(5^x - x^5)\ln 5} \cdot \left(5^x - x^5\right)' = \frac{5^x \ln 5 - 5x^4}{(5^x - x^5)\ln 5}$$

$\boxed{\left(\log_a (\text{nešto})\right)' = \frac{1}{\text{nešto} \cdot \ln a} \cdot (\text{nešto})'}$ $\boxed{(\log_a x)' = \frac{1}{x \ln a}}$

Zadatak 12

$\left((\text{nešto})^n\right)' = n(\text{nešto})^{n-1} \cdot (\text{nešto})'$ $(x^n)' = nx^{n-1}$

Odredite derivacije funkcija $f(x) = \ln 5x$, $g(x) = \ln x^5$ *i* $h(x) = \ln^5 x$.

Rješenje

$$f(x) = \ln 5x$$
$$f'(x) = \frac{1}{5x} \cdot (5x)'$$
$$f'(x) = \frac{1}{5x} \cdot 5$$
$$f'(x) = \frac{1}{x}$$

$$g(x) = \ln x^5$$
$$g'(x) = \frac{1}{x^5} \cdot (x^5)'$$
$$g'(x) = \frac{1}{x^5} \cdot 5x^4$$
$$g'(x) = \frac{5}{x}$$

$$h(x) = \ln^5 x$$
$$h(x) = (\ln x)^5$$
$$h'(x) = 5\,(\ln x)^4 \cdot (\ln x)'$$
$$h'(x) = 5\,(\ln x)^4 \cdot \frac{1}{x}$$
$$h'(x) = \frac{5}{x} \ln^4 x$$

$\left(\ln(\text{nešto})\right)' = \frac{1}{\text{nešto}} \cdot (\text{nešto})'$ $(\ln x)' = \frac{1}{x}$ $\ln^k x = (\ln x)^k$

Zadatak 13

Odredite derivaciju funkcije $y = \ln \frac{x-1}{x+1}$.

$(\ln x)' = \frac{1}{x}$

Rješenje

$\left(\ln(\text{nešto})\right)' = \frac{1}{\text{nešto}} \cdot (\text{nešto})'$

$$y' = \frac{1}{\frac{x-1}{x+1}} \cdot \left(\frac{x-1}{x+1}\right)' =$$
$$= \frac{x+1}{x-1} \cdot \frac{(x-1)' \cdot (x+1) - (x-1) \cdot (x+1)'}{(x+1)^2} =$$
$$= \frac{x+1}{x-1} \cdot \frac{1 \cdot (x+1) - (x-1) \cdot 1}{(x+1)^2} = \frac{2}{(x-1)(x+1)} = \frac{2}{x^2-1}$$

$\left(\frac{u}{v}\right)'(x) = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v(x)^2}$

Zadatak 14

Odredite derivaciju funkcije $y = \sqrt{\sin(x^2-1)}$.

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

$(\sin x)' = \cos x$

Rješenje

$$y' = \frac{1}{2\sqrt{\sin(x^2-1)}} \cdot \left(\sin(x^2-1)\right)' =$$
$$= \frac{1}{2\sqrt{\sin(x^2-1)}} \cdot \cos(x^2-1) \cdot (x^2-1)' =$$
$$= \frac{2x\cos(x^2-1)}{2\sqrt{\sin(x^2-1)}} = \frac{x\cos(x^2-1)}{\sqrt{\sin(x^2-1)}}$$

$\left(\sqrt{\text{nešto}}\right)' = \frac{1}{2\sqrt{\text{nešto}}} \cdot (\text{nešto})'$

$\left(\sin(\text{nešto})\right)' = \cos(\text{nešto}) \cdot (\text{nešto})'$

Zadatak 15

Odredite četvrtu derivaciju funkcije $f(x) = \ln(3x+1)$.

Rješenje

$\left(\ln(\text{nešto})\right)' = \frac{1}{\text{nešto}} \cdot (\text{nešto})'$ $(\ln x)' = \frac{1}{x}$

- Prva derivacija

$$f'(x) = \frac{1}{3x+1} \cdot (3x+1)' = \frac{1}{3x+1} \cdot 3 = \frac{3}{3x+1}$$
$$f'(x) = 3 \cdot (3x+1)^{-1}$$

- Druga derivacija

$$f''(x) = 3 \cdot (-1) \cdot (3x+1)^{-2} \cdot (3x+1)' = -3 \cdot (3x+1)^{-2} \cdot 3$$
$$f''(x) = -9 \cdot (3x+1)^{-2}$$

$\left((\text{nešto})^n\right)' = n(\text{nešto})^{n-1} \cdot (\text{nešto})'$ $(x^n)' = nx^{n-1}$

$f''(x) = -9 \cdot (3x+1)^{-2}$

- Treća derivacija

$$f'''(x) = -9 \cdot (-2) \cdot (3x+1)^{-3} \cdot (3x+1)' = 18 \cdot (3x+1)^{-3} \cdot 3$$

$$f'''(x) = 54 \cdot (3x+1)^{-3}$$

- Četvrta derivacija

$$f^{(4)}(x) = 54 \cdot (-3) \cdot (3x+1)^{-4} \cdot (3x+1)' = -162 \cdot (3x+1)^{-4} \cdot 3$$

$$f^{(4)}(x) = -486 \cdot (3x+1)^{-4}$$

$$\boxed{f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'}$$

$\big((\text{nešto})^n\big)' = n(\text{nešto})^{n-1} \cdot (\text{nešto})'$ $\quad$ $(x^n)' = nx^{n-1}$

Zadatak 16

Odredite jednadžbu tangente na graf funkcije $y = \ln(5-4x)$ u točki s apscisom 1. Odredite duljinu odsječka dobivene tangente između koordinatnih osi.

Rješenje

- Jednadžba tangente na graf funkcije $y = f(x)$ u točki $T_0(x_0, y_0)$

$$t \dots \; y - y_0 = k_t \cdot (x - x_0)$$

- Pritom je $y_0 = f(x_0)$ i $k_t = f'(x_0)$.

$y = \ln(5-4x)$

- Znamo da je $x_0 = 1$.

$$y_0 = \ln(5 - 4 \cdot 1) = \ln 1 = 0$$

Točka: $T_0(1, 0)$

- Derivacija funkcije

$$y' = \frac{1}{5-4x} \cdot (5-4x)' = \frac{1}{5-4x} \cdot (-4) = \frac{-4}{5-4x}$$

- Koeficijent smjera tangente

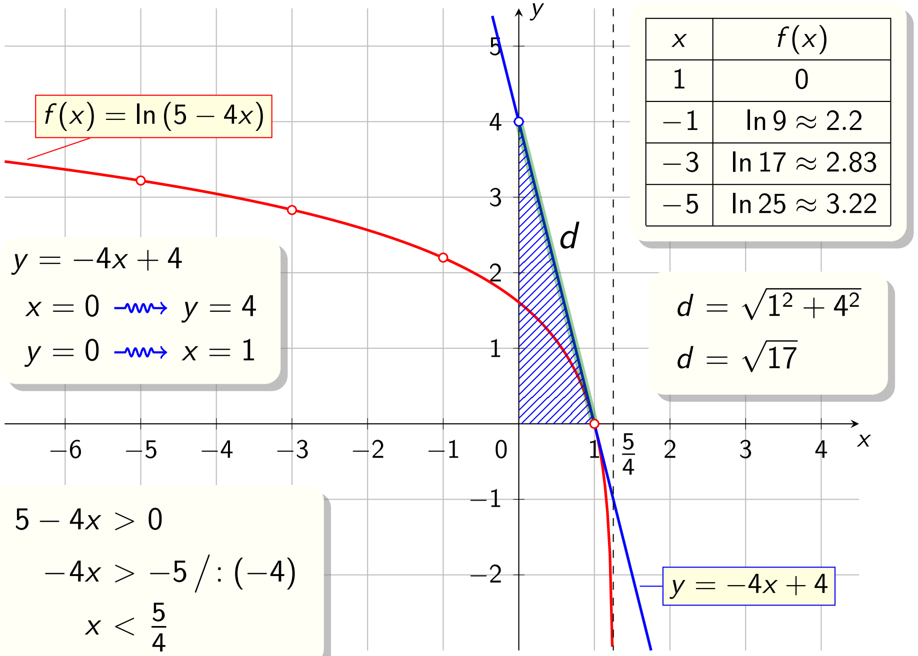
$$k_t = y'(1) = \frac{-4}{5 - 4 \cdot 1} = -4$$

$\big(\ln(\text{nešto})\big)' = \dfrac{1}{\text{nešto}} \cdot (\text{nešto})'$ $\quad$ $(\ln x)' = \dfrac{1}{x}$

$x_0 = 1$ $\quad$ $y_0 = 0$ $\quad$ $k_t = -4$

- Jednadžba tangente

$$y - y_0 = k_t \cdot (x - x_0)$$
$$y - 0 = -4 \cdot (x - 1)$$
$$\boxed{y = -4x + 4}$$



x	$f(x)$
1	0
-1	$\ln 9 \approx 2.2$
-3	$\ln 17 \approx 2.83$
-5	$\ln 25 \approx 3.22$

$y = -4x + 4$

$x = 0 \rightsquigarrow y = 4$

$y = 0 \rightsquigarrow x = 1$

$$d = \sqrt{1^2 + 4^2}$$

$$d = \sqrt{17}$$

$$5 - 4x > 0$$

$$-4x > -5 \;/ : (-4)$$

$$x < \frac{5}{4}$$