Sigma zapis, binomni teorem i matematička indukcija

Matematika za ekonomiste 1

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Σ notacija

$\sum$ grčko slovo sigma

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{\alpha=1}^{n} a_\alpha$$

Zadatak 1

Napišite sljedeće izraze pomoću Σ notacije:

a) $10 + 20 + 40 + \cdots + 5 \cdot 2^n$

b) $1^2 \cdot 2 + 2^2 \cdot 3 + 3^2 \cdot 4 + \cdots + n^2 \cdot (n+1)$

c) $3^2 \cdot 4 + 4^2 \cdot 5 + 5^2 \cdot 6 + \cdots + (n-2)^2 \cdot (n-1)$

d) $-4 - 8 - 12 - \cdots - 4k$

Rješenje

a)

$$\underbrace{10}_{5\cdot 2^1} + \underbrace{20}_{5\cdot 2^2} + \underbrace{40}_{5\cdot 2^3} + \cdots + 5 \cdot 2^n = \sum_{i=1}^{n} 5 \cdot 2^i$$

b)

$$1^2 \cdot 2 + 2^2 \cdot 3 + 3^2 \cdot 4 + \cdots + n^2 \cdot (n+1) = \sum_{k=1}^{n} k^2(k+1)$$

c)

$$3^2 \cdot 4 + 4^2 \cdot 5 + 5^2 \cdot 6 + \cdots + (n-2)^2 \cdot (n-1) = \sum_{k=3}^{n-2} k^2(k+1)$$

$$3^2 \cdot 4 + 4^2 \cdot 5 + 5^2 \cdot 6 + \cdots + (n-2)^2 \cdot (n-1) = \sum_{k=5}^{n} (k-2)^2(k-1)$$

d)

$$-4-8-12-\cdots-4k = -4+(-8)+(-12)+\cdots+(-4k) =$$

$$= \sum_{j=1}^{k}(-4j)$$

Zadatak 2

Napišite sljedeće izraze bez Σ notacije:

a) $\sum_{\alpha=3}^{5}\alpha^2$

b) $\sum_{i=1}^{n}2^{i+2}$

c) $\sum_{k=5}^{n+2}(2k-1)$

d) $\sum_{j=2}^{n-1}a_k$

Rješenje

a) $$\sum_{\alpha=3}^{5}\alpha^2 = \underset{\alpha=3}{3^2} + \underset{\alpha=4}{4^2} + \underset{\alpha=5}{5^2} = 50$$

b) $$\sum_{i=1}^{n}2^{i+2} = \underset{i=1}{2^3} + \underset{i=2}{2^4} + \underset{i=3}{2^5} + \cdots + \underset{i=n}{2^{n+2}} = 8+16+32+\cdots+2^{n+2}$$

c) $$\sum_{k=5}^{n+2}(2k-1) = \underset{k=5}{9} + \underset{k=6}{11} + \underset{k=7}{13} + \cdots + \big(2\underset{k=n+2}{(n+2)}-1\big) =$$

$$= 9+11+13+\cdots+(2n+3)$$

$$\boxed{\sum_{j=3}^{n}(2j+3)}$$

Napomena

$$\sum_{k=5}^{n+2}2k-1 = \underset{k=5}{10} + \underset{k=6}{12} + \underset{k=7}{14} + \cdots + \underset{k=n+2}{(2n+4)} - 1$$

d) $\displaystyle\sum_{j=2}^{n-1} a_k = \overbrace{\underset{j=2}{a_k} + \underset{j=3}{a_k} + \underset{j=4}{a_k} + \cdots + \underset{j=n-1}{a_k}}^{n-2} = (n-2)a_k$

n faktorijela

$$n! = 1 \cdot 2 \cdot 3 \cdots n, \quad n \in \mathbb{N}$$

$$5! = \underbrace{\overbrace{1 \cdot 2 \cdot 3}^{3!} \cdot 4}_{4!} \cdot 5 = 120$$

$$n! = (n-1)! \cdot n \qquad\qquad n! = (n-2)! \cdot (n-1) \cdot n$$

- Po dogovoru je $0! = 1$.

Binomni koeficijent

n povrh *k*

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} \qquad\qquad \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots k}$$

Primjer

$$\binom{6}{4} = \frac{6!}{4! \cdot (6-4)!} = \frac{6!}{4! \cdot 2!} = \frac{4! \cdot 5 \cdot 6}{4! \cdot 2} = 15$$

$$\binom{6}{4} = \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} = 15$$

$$\boxed{\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}} \qquad\qquad \boxed{\binom{n}{k} = \frac{n \cdot (n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots k}}$$

$$\binom{n}{0} = \frac{n!}{0! \cdot (n-0)!} = \frac{n!}{1 \cdot n!} = 1$$

$$\binom{n}{1} = \frac{n!}{1! \cdot (n-1)!} = \frac{(n-1)! \cdot n}{(n-1)!} = n$$

$$\binom{n}{1} = \frac{n}{1} = n$$

$$\binom{n}{n} = \frac{n!}{n! \cdot (n-n)!} = \frac{n!}{n! \cdot 0!} = 1$$

$$\binom{n}{n} = \frac{n \cdot (n-1) \cdots 2 \cdot 1}{1 \cdot 2 \cdots (n-1) \cdot n} = 1$$

Svojstvo simetrije

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{6}{4} = \binom{6}{6-4} = \binom{6}{2} = \frac{6\cdot 5}{1\cdot 2} = 15$$

$$\binom{100}{97} = \binom{100}{100-97} = \binom{100}{3} = \frac{100\cdot 99\cdot 98}{1\cdot 2\cdot 3} = 161\,700$$

Pascalov trokut

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Binomni teorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \qquad n \in \mathbb{N}$$

$$(a+b)^n = \underbrace{\binom{n}{0} a^n b^0}_{k=0} + \underbrace{\binom{n}{1} a^{n-1} b^1}_{k=1} + \underbrace{\binom{n}{2} a^{n-2} b^2}_{k=2} + \cdots + \underbrace{\binom{n}{n} a^0 b^n}_{k=n}$$

$$(a+b)^2 = \sum_{k=0}^{2} \binom{2}{k} a^{2-k} b^k = \underbrace{\binom{2}{0} a^2 b^0}_{k=0} + \underbrace{\binom{2}{1} a^1 b^1}_{k=1} + \underbrace{\binom{2}{2} a^0 b^2}_{k=2} =$$

$$= a^2 + 2ab + b^2$$

Zadatak 3

Pomoću binomnog teorema raspišite i sredite binom $\left(\sqrt[3]{x} + x^2\right)^4$.

Rješenje

$$\left(\sqrt[3]{x} + x^2\right)^4 = \binom{4}{0} \sqrt[3]{x}^{\,4} \left(x^2\right)^0 + \binom{4}{1} \sqrt[3]{x}^{\,3} \left(x^2\right)^1 +$$

$$+ \binom{4}{2} \sqrt[3]{x}^{\,2} \left(x^2\right)^2 + \binom{4}{3} \sqrt[3]{x}^{\,1} \left(x^2\right)^3 + \binom{4}{4} \sqrt[3]{x}^{\,0} \left(x^2\right)^4 =$$

$$= 1\cdot x^{\frac{4}{3}} \cdot 1 + 4\cdot x \cdot x^2 + 6\cdot x^{\frac{2}{3}} \cdot x^4 + 4\cdot x^{\frac{1}{3}} \cdot x^6 + 1\cdot 1\cdot x^8 =$$

$$= x^{\frac{4}{3}} + 4x^3 + 6x^{\frac{14}{3}} + 4x^{\frac{19}{3}} + x^8$$

$\boxed{\sqrt[n]{x^m} = x^{\frac{m}{n}}}$ $\boxed{\left(x^m\right)^n = x^{mn}}$ $\boxed{x^m \cdot x^n = x^{m+n}}$

Domaća zadaća

Zadatak

Pomoću binomnog teorema raspišite i sredite binom $\left(\sqrt[3]{x}-x^2\right)^4$.

Rješenje

$$\left(\sqrt[3]{x}+(-x^2)\right)^4=\binom{4}{0}\sqrt[3]{x}^{\,4}\left(-x^2\right)^0+\binom{4}{1}\sqrt[3]{x}^{\,3}\left(-x^2\right)^1+$$

$$+\binom{4}{2}\sqrt[3]{x}^{\,2}\left(-x^2\right)^2+\binom{4}{3}\sqrt[3]{x}^{\,1}\left(-x^2\right)^3+\binom{4}{4}\sqrt[3]{x}^{\,0}\left(-x^2\right)^4=$$

$$=1\cdot x^{\frac{4}{3}}\cdot 1-4\cdot x\cdot x^2+6\cdot x^{\frac{2}{3}}\cdot x^4-4\cdot x^{\frac{1}{3}}\cdot x^6+1\cdot 1\cdot x^8=$$

$$=x^{\frac{4}{3}}-4x^3+6x^{\frac{14}{3}}-4x^{\frac{19}{3}}+x^8$$

Zadatak 4

$\boxed{x^m\cdot x^n=x^{m+n}}$ $\boxed{\left(x^m\right)^n=x^{mn}}$ $\boxed{(xy)^n=x^ny^n}$

Pomoću binomnog teorema raspišite i sredite binom $\left(x^{\frac{3}{2}}y+y^{-1}\right)^5$.

Rješenje

$$\left(x^{\frac{3}{2}}y+y^{-1}\right)^5=\binom{5}{0}\left(x^{\frac{3}{2}}y\right)^5\left(y^{-1}\right)^0+\binom{5}{1}\left(x^{\frac{3}{2}}y\right)^4\left(y^{-1}\right)^1+$$

$$+\binom{5}{2}\left(x^{\frac{3}{2}}y\right)^3\left(y^{-1}\right)^2+\binom{5}{3}\left(x^{\frac{3}{2}}y\right)^2\left(y^{-1}\right)^3+\binom{5}{4}\left(x^{\frac{3}{2}}y\right)^1\left(y^{-1}\right)^4+$$

$$+\binom{5}{5}\left(x^{\frac{3}{2}}y\right)^0\left(y^{-1}\right)^5=1\cdot x^{\frac{15}{2}}y^5\cdot 1+5\cdot x^6y^4\cdot y^{-1}+10\cdot x^{\frac{9}{2}}y^3\cdot y^{-2}+$$

$$+10\cdot x^3y^2\cdot y^{-3}+5\cdot x^{\frac{3}{2}}y\cdot y^{-4}+1\cdot 1\cdot y^{-5}=$$

$$=x^{\frac{15}{2}}y^5+5x^6y^3+10x^{\frac{9}{2}}y+10x^3y^{-1}+5x^{\frac{3}{2}}y^{-3}+y^{-5}$$

Matematička indukcija

Neka je $P(n)$ tvrdnja koja ovisi o $n\in\mathbb{N}$.

- $P(1)$ je istinita tvrdnja.
- Ako je $P(k)$ istinita tvrdnja, tada je $P(k+1)$ istinita tvrdnja.

$$P(1)\Rightarrow P(2)\Rightarrow P(3)\Rightarrow P(4)\Rightarrow P(5)\Rightarrow P(6)\Rightarrow\cdots$$

Zaključak

$P(n)$ je tvrdnja koja vrijedi za sve prirodne brojeve.

Zadatak 5

Dokažite matematičkom indukcijom da za svaki $n\in\mathbb{N}$ *vrijedi*

$$4+20+48+\cdots+2n(3n-1)=2n^2(n+1).$$

Rješenje

- Baza indukcije: $n=1$

$$4=2\cdot 1^2\cdot(1+1)$$
$$4=4$$

- Korak indukcije

  Pretpostavimo da tvrdnja vrijedi za neki $n\in\mathbb{N}$, tj. da vrijedi

$$4+20+48+\cdots+2n(3n-1)=2n^2(n+1).$$

$$\boxed{4 + 20 + 48 + \cdots + 2n(3n-1) = 2n^2(n+1)}$$

desna strana za $n+1$

$$2(n+1)^2\big((n+1)+1\big)$$

$$\boxed{2(n+1)^2(n+2)}$$

Želimo dokazati da tvrdnja vrijedi za sljedeći prirodni broj $n+1$.

$$\underbrace{4 + 20 + 48 + \cdots + 2n(3n-1)}_{\text{pretpostavka indukcije}} + 2(n+1)\big(3(n+1)-1\big) =$$

$$= 2n^2(n+1) + 2(n+1)(3n+2) =$$

$$= 2(n+1)\big(n^2 + (3n+2)\big) = 2(n+1)(n^2 + 3n + 2) =$$

$$= 2(n+1)\cdot(n+1)(n+2) = 2(n+1)^2(n+2)$$

$$\boxed{ax^2 + bx + c = a(x-x_1)(x-x_2)}$$